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Linearized stability in periodic functional differential equations with state-dependent delays☆

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Abstract

In this paper, we study stability of periodic solutions of a class of nonlinear functional differential equations (FDEs) with state-dependent delays using the method of linearization. We show that a periodic solution of the nonlinear FDE is exponentially stable, if the zero solution of an associated linear periodic linear homogeneous FDE is exponentially stable.

Keywords: Linearization; State-dependent delay; Stability; Periodic solution

1. Introduction

Functional differential equation with state-dependent delays (sd-FDEs) appear frequently in applications as model equations (see, e.g., [1,2,4,19]) and the study of such equations is an active research area (see, e.g., [7,14–16]). Stability of the solution is one of the most important qualitative property of a model. There are many papers which give sufficient conditions for the stability of the trivial (zero) solution in sd-FDEs (see, e.g., [8,21,22]).

For nonlinear equations the method of linearization is a standard tool in stability investigations, but for sd-FDEs there are many technical problems with it (see, e.g., [3,10,12,14]). Linearization theorems for obtaining stability of the zero or constant equilibriums were given in [5,11,13] for various classes of sd-FDEs. In this paper, we extend these results for periodic solutions of a class of nonlinear sd-FDEs (see Theorem 2.5 below). Our results were motivated in [16], where the existence

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of such result was conjectured, and extensive numerical investigation of stability of constant and periodic solutions of sd-FDEs was given.

For results concerning the existence of a periodic solutions of sd-FDEs we refer the interested reader to [6,17,18,20].

2. Main results

Consider the nonlinear state-dependent delay system

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t, x_t))), \quad t \geqslant 0 \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \quad t \in [-r, 0]. \tag{2.2}$$

Here and later on x_t denotes the solution segment function, i.e., $x_t(s) = x(t+s)$ for $s \in [-r, 0]$. The Banach-space of continuous functions $\psi : [-r, 0] \to \mathbb{R}^n$ with the supremum norm $\|\psi\| = \max\{|\psi(s)| : s \in [-r, 0]\}$ is denoted by C. A closed neighborhood with radius ϱ of a set A in a Banach-space X is denoted by $B_X(A; \varrho) = \{x \in X : |x - a|_X \leqslant \varrho$ for some $a \in A\}$. We use $|\cdot|$ for any fixed norm on $\mathbb{R}^n$ and for the corresponding induced matrix norm on $\mathbb{R}^{n \times n}$, as well. $\mathscr{L}(C, \mathbb{R})$ denotes the Banach-space of bounded linear functionals on C with the norm $|\cdot|_{\mathscr{L}(C,\mathbb{R})}$.

We assume the following conditions throughout the paper:

(H1) $f : [0, \infty) \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuously differentiable, where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$, and let f be T-periodic, i.e.,

$$f(t, u, v) = f(t + T, u, v), \quad t \geqslant 0, \ u \in \Omega_1, \ v \in \Omega_2,$$

(H2) (i) $\tau : [0, \infty) \times \Omega_3 \to [0, r]$ is continuously differentiable, where Ω_3 is an open subset of C, and τ is T-periodic, i.e.,

$$\tau(t, \psi) = \tau(t + T, \psi), \quad t \geqslant 0, \ \psi \in \Omega_3,$$

(ii) τ is locally Lipschitz-continuous in the following sense: for every bounded and closed subset M of C there exists a constant $L_1 = L_1(M) \geqslant 0$ such that

$$|\tau(t, \psi) - \tau(t, \tilde{\psi})| \leqslant L_1 \|\psi - \tilde{\psi}\|, \quad t \in [0, T], \ \psi, \tilde{\psi} \in M,$$

(iii) $D_2\tau$ is locally Lipschitz-continuous in the following sense: for every bounded and closed subset M of C there exists a constant $L_2 = L_2(M) \geqslant 0$ such that

$$|D_2\tau(t, \psi) - D_2\tau(t, \tilde{\psi})|_{\mathscr{L}(C,\mathbb{R})} \leqslant L_2 \|\psi - \tilde{\psi}\|, \quad t \in [0, T], \ \psi, \tilde{\psi} \in M.$$

Let $\bar{x} : [-r, \infty) \to \mathbb{R}^n$ be a T-periodic solution of (2.1). The restriction of $\bar{x}$ to the interval $[-r, 0]$ is denoted by $\bar{\varphi}$, i.e., $\bar{x}$ is the solution of (2.1) and (2.2) corresponding to initial function $\bar{\varphi}$. It is assumed that $\bar{\varphi}$ and $\bar{x}$ are fixed throughout this paper. Since $\bar{x}$ is a solution of (2.1), the continuity of f and τ imply that $\dot{\bar{x}}$ is continuous on $[0, \infty)$, therefore, $\bar{x}$ is continuously differentiable on $[-r, \infty)$, as well.