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Generalizations of orthogonal polynomials ☆

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This paper is dedicated to Olav Njåstad on the occasion of his 70th birthday

Abstract

We give a survey of recent generalizations of orthogonal polynomials. That includes multidimensional (matrix and vector orthogonal polynomials) and multivariate versions, multipole (orthogonal rational functions) variants, and extensions of the orthogonality conditions (multiple orthogonality). Most of these generalizations are inspired by the applications in which they are applied. We also give a glimpse of these applications, which are usually generalizations

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of applications where classical orthogonal polynomials also play a fundamental role: moment problems, numerical quadrature, rational approximation, linear algebra, recurrence relations, and random matrices.

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1. Introduction

Since the fundamental work of Szegő [48], orthogonal polynomials have been an essential tool in the analysis of basic problems in mathematics and engineering. For example moment problems, numerical quadrature, rational and polynomial approximation and interpolation, linear algebra, and all the direct or indirect applications of these techniques in engineering and applied problems, they are all indebted to the basic properties of orthogonal polynomials.

Obviously, if we want to discuss *orthogonal* polynomials, the first thing we need is an inner product defined on the space of polynomials. There are several formalizations of this concept. For example, one can define a positive definite Hermitian linear functional $M[\cdot]$ on the space of polynomials. This means the following. Let Π_n be the space of polynomials of degree at most n and Π the space of all polynomials. The dual space of Π_n is Π_{n*}, namely the space of all linear functionals. With respect to a set of basis functions $\{B_0, B_1, \ldots, B_n\}$ that span Π_n for $n = 0, 1, \ldots$, a polynomial has a uniquely defined set of coefficients, representing this polynomial. Thus, given a nested basis of Π, we can identify the space of complex polynomials Π_n with the space of its coefficients, i.e., with $\mathbb{C}^{(n+1)\times 1}$ of complex $(n+1)\times 1$ column vectors.

Suppose the dual space is spanned by a sequence of basic linear functionals $\{L_k\}_{k=0}^{\infty}$, thus $\Pi_{n*} = \operatorname{span}\{L_0, L_1, \ldots, L_n\}$ for $n = 0, 1, 2, \ldots$. Then the dual subspace Π_{n*} can be identified with $\mathbb{C}^{1\times(n+1)}$, the space of complex $1\times(n+1)$ row vectors. Now, given a sequence of linear functionals $\{L_k\}_{k=0}^{\infty}$, we say that a sequence of polynomials $\{P_k\}_{k=0}^{\infty}$ with $P_k \in \Pi_k$, is orthonormal with respect to the sequence of linear functionals $\{L_k\}_{k=0}^{\infty}$ with $L_k \in \Pi_{k*}$, if

$$L_k(P_l) = \delta_{kl}, \quad k, l = 0, 1, 2 \ldots .$$

Hereby we have to assure some non-degeneracy, which means that the moment matrix of the system is Hermitian positive definite. This moment matrix is defined as follows. Consider the basis $B_0, B_1, \ldots$ in Π and a basis $L_0, L_1, \ldots$ for the dual space Π_*, then the moment matrix is the infinite matrix

$$M = \begin{bmatrix} m_{00} & m_{01} & m_{02} & \cdots \\ m_{10} & m_{11} & m_{12} & \cdots \\ m_{20} & m_{21} & m_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \text{with} \quad m_{ij} = L_i(B_j).$$

It is Hermitian positive definite if $M_{kk} = [m_{ij}]_{i,j=0}^{k}$ is Hermitian positive definite for all $k = 0, 1, \ldots$.

In some formal generalizations, positive definiteness may not be necessary; a nondegeneracy condition is then sufficient (all the leading principal submatrices are nonsingular rather than positive definite). In other applications it is not even really necessary to impose this nondegeneracy condition, and in that case