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Weak classical orthogonal polynomials in two variables☆

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Abstract

Orthogonal polynomials in two variables constitute a very old subject in approximation theory. Usually they are studied as solutions of second-order partial differential equations. In this work, we study two-variable orthogonal polynomials associated with a moment functional satisfying the two-variable analogue of the Pearson differential equation. From this approach, we derive the extension of some of the usual characterizations of the classical orthogonal polynomials in one variable.

MSC: 33C50

Keywords: Orthogonal polynomials in two variables; Classical orthogonal polynomials

1. Introduction

One of the most important characterizations of the classical orthogonal polynomials in one variable was given by Bochner in 1929 (see [1]). Let $\{P_n\}_n$ be a sequence of classical orthogonal polynomials (Hermite, Laguerre, Jacobi or Bessel), then $P_n(x)$ is solution of the second-order differential equation

$$\phi(x)y'' + \psi(x)y' = \lambda_n y, \tag{1}$$

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where $\phi(x)$ and $\psi(x)$ are fixed polynomials of degree $\leqslant 2$, and $\leqslant 1$, respectively, and λ_n is a real number depending on the degree of the polynomial solution. Conversely, given a second-order differential equation as (1), Bochner showed that, up to a linear change of the variable with complex coefficients, the only sequences of orthogonal polynomials satisfying (1) are those of Hermite, Laguerre, Jacobi and Bessel.

One of the main features in the study of orthogonal polynomials in two variables is the fact that, for $n > 0$, the set of the polynomials of total degree n orthogonal to all the polynomials of lower degree, is a linear space of dimension $n + 1$.

Using a vector representation, Kowalski (see [9,10]) proved that orthogonal polynomials in several variables can be characterized by a three term recurrence relation with matrix coefficients. Xu (see [2,17]) gave another formulation of the recurrence relations and gave a simpler proof of Kowalski's results. Moreover, he obtained a Christoffel–Darboux type formula, properties of the zeros and so on. However, the concept of classical character for a family of orthogonal polynomials in two variables is not very clear in the literature about this subject.

The two-variable analogous of the Bochner equation (1) was studied by Krall and Sheffer in 1967 (see [11]). In fact, they characterized the classical orthogonal polynomials in two variables as the polynomial solutions of the second-order partial differential equation

$$\begin{aligned}(ax^2 + d_1x + e_1y + f_1)w_{xx} &+ (2axy + d_2x + e_2y + f_2)w_{xy} \\ &+ (ay^2 + d_3x + e_3y + f_3)w_{yy} + (gx + h_1)w_x + (gy + h_2)w_y = \lambda_n w, \end{aligned} \tag{2}$$

where $\lambda_n = an(n-1) + gn$.

This equation depends only on the total degree of the polynomial solution and, therefore, all the polynomials of total degree n satisfy the same equation.

These authors obtain nine types of classical orthogonal polynomials in two variables, depending on the canonical shapes of the polynomial coefficients in (2). In this classification appear, among others, the Jacobi polynomials over the simplex, the Laguerre tensor product polynomials, and the Hermite tensor product polynomials.

However, $P_{h,k}^{(\alpha,\beta,\hat{\alpha},\hat{\beta})}(x,y) = P_h^{(\alpha,\beta)}(x)P_k^{(\hat{\alpha},\hat{\beta})}(y)$, the tensor product of Jacobi polynomials, orthogonal on $[-1,1]\times[-1,1]$ with respect to the weight function $w(x,y) = (1-x)^\alpha(1+x)^\beta(1-y)^{\hat{\alpha}}(1+y)^{\hat{\beta}}$, $\alpha, \beta, \hat{\alpha}, \hat{\beta} > -1$, is not a classical family according to the classification given in [11]. The explanation of this situation is that the tensor Jacobi polynomials satisfy the second-order partial differential equation

$$\begin{aligned}(1-x^2)w_{xx} &+ (1-y^2)w_{yy} + [\beta - \alpha - (\alpha+\beta+2)x]\, w_x \\ &+ [\hat{\beta} - \hat{\alpha} - (\hat{\alpha}+\hat{\beta}+2)y]w_y = \lambda_{h,k} w,\end{aligned}$$

where the coefficient of the term without derivatives depends on the partial degrees of the polynomial solution, and consequently, it is not of type (2).

The aim of this work is to extend the concept of classical orthogonal polynomials in two variables as the solutions of a matrix second-order partial differential equation involving matrix polynomial coefficients, the usual gradient operator ∇, and the divergence operator div.

The structure of the paper is as follows. First, we introduce the necessary definitions and basic tools for the rest of the paper. For more details, see [2,16]. In Section 3, we introduce the orthogonal polynomials in two variables associated with a moment functional u satisfying the two-variable analogue of the Pearson