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How to add a non-integer number of terms, and how to produce unusual infinite summations

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Abstract

Sums of the form $\sum_{v=1}^{x} f(v)$ are defined traditionally only when the number of terms x is a positive integer or ∞. We propose a natural way to extend this definition to the case when x is a (rather arbitrary) real or complex number ("fractional sums"). This generalizes known special cases like the interpolation of the factorial by the Γ function, or Euler's little-known formula

$$\sum_{v=1}^{-1/2} \frac{1}{v} = -2 \ln 2.$$

After giving the fundamental definition, we generalize several algebraic identities (such as the geometric series) to the case with a non-integer number of terms.

We use these ideas to derive a number of unusual infinite sums, products and limits, such as

$$\lim_{n\to\infty} \left((2n)^{-n^2/2-n/4}\, \mathrm{e}^{-n/8} \prod_{v=1}^{2n} v^{(-1)^v v^2/4} \right) = \mathrm{e}^{7\zeta(3)/16\pi^2}.$$

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1. Introduction

In this note, we determine a number of unusual infinite sums, products, or limits, such as

$$2\sum_{\nu\geqslant 1 \text{ odd}} \frac{1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots++\frac{1}{\nu^s}}{\nu^s} = \zeta(2s)(1-2^{-2s}) + (\zeta(s)(1-2^{-s}))^2 \tag{1}$$

for every real $s > 1$ (using the Riemann ζ function), or, for every $a > 0$,

$$\prod_{\nu=1}^{\infty} \frac{1}{\mathrm{e}}\left(1+\frac{1}{a\nu}\right)^{a\nu+1/2} = \sqrt{\frac{\Gamma(1+1/a)}{2\pi}} \exp\left[\frac{1}{2}\left(1+\frac{1}{a}\right) - a\left(\zeta'\left(-1, 1+\frac{1}{a}\right) - \zeta'(-1)\right)\right]. \tag{2}$$

Examples of limits include

$$\lim_{n\to\infty}\left((2n)^{-n^2/2-n/4}\,\mathrm{e}^{-n/8}\prod_{\nu=1}^{2n}\nu^{(-1)^\nu \nu^2/4}\right) = \mathrm{e}^{7\zeta(3)/16\pi^2} \tag{3}$$

or

$$\lim_{n\to\infty}\left[-\frac{5}{16}\ln n\ln(n!) + \frac{1}{16}\ln(n+2)\ln((n+2)!) - \frac{1}{4}\ln(n+1)\ln((n+1)!) + \sum_{\nu=1}^{2n}(-1)^\nu \ln\frac{\nu}{2}\ln\left(\left(\frac{\nu}{2}\right)!\right)\right] = \frac{\gamma^2}{4} + \frac{\gamma_1}{2} - \frac{\pi^2}{48} + \frac{\ln^2 2}{2} - \frac{\ln^2\pi}{8}, \tag{4}$$

where γ and γ_1 are the Euler–Mascheroni and Stieltjes constants

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} - \ln n\right) = .577215\ldots, \tag{5}$$

$$\gamma_1 = \lim_{n\to\infty}\left(\frac{\ln 1}{1}+\frac{\ln 2}{2}+\cdots+\frac{\ln n}{n} - \frac{\ln^2 n}{2}\right) = -.072815\ldots\;.$$

We try to prove (or sketch the proofs of) all these identities in this note. At the end, we speculate about a short and unusual proof of the following identity from [4] (for $b \in \mathbb{R}$):

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\sqrt{b^2+\pi^2 n^2} = \frac{\pi^2}{4}\left(\frac{\sin\, b}{b} - \frac{\cos\, b}{3}\right). \tag{6}$$

These identities were derived by a method to evaluate sums in which the number of terms is no longer an integer, but almost any real or complex number. While this method has been mentioned by Euler or Ramanujan, we are not aware of any systematic attempts to work it out.