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# Multiple little $q$-Jacobi polynomials ☆

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**Abstract**

We introduce two kinds of multiple little $q$-Jacobi polynomials $p_{\vec{n}}$ with multi-index $\vec{n} = (n_1, n_2, \ldots, n_r)$ and degree $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ by imposing orthogonality conditions with respect to $r$ discrete little $q$-Jacobi measures on the exponential lattice $\{q^k, k = 0, 1, 2, 3, \ldots\}$, where $0 < q < 1$. We show that these multiple little $q$-Jacobi polynomials have useful $q$-difference properties, such as a Rodrigues formula (consisting of a product of $r$ difference operators). Some properties of the zeros of these polynomials and some asymptotic properties will be given as well.


*Keywords:* $q$-Jacobi polynomials; Basis hypergeometric polynomials; Multiple orthogonal polynomials

## 1. Little $q$-Jacobi polynomials

Little $q$-Jacobi polynomials are orthogonal polynomials on the exponential lattice $\{q^k,\ k=0, 1, 2, \ldots\}$, where $0 < q < 1$. In order to express the orthogonality relations, we will use the $q$-integral

$$\int_0^1 f(x)\, d_q x = (1-q) \sum_{k=0}^{\infty} q^k f(q^k) \tag{1.1}$$

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(see, e.g., [2, Section 10.1; 5, Section 1.11]) where *f* is a function on [0, 1] which is continuous at 0. The orthogonality is given by

$$\int_0^1 p_n(x;\alpha,\beta\,|\,q)x^k w(x;\alpha,\beta\,|\,q)\,\mathrm{d}_q x = 0, \quad k = 0, 1, \ldots, n-1, \tag{1.2}$$

where

$$w(x;a,b\,|\,q) = \frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\,x^\alpha. \tag{1.3}$$

We have used the notation

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

In order that the $q$-integral of $w$ is finite, we need to impose the restrictions $\alpha, \beta > -1$. The orthogonality conditions (1.2) determine the polynomials $p_n(x;\alpha,\beta\,|\,q)$ up to a multiplicative factor. In this paper, we will always use monic polynomials and these are uniquely determined by the orthogonality conditions. The $q$-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|, |q| < 1 \tag{1.4}$$

(see, e.g., [2, Section 10.2; 5, Section 1.3]) implies that

$$\lim_{q\to 1} w(x;\alpha,\beta\,|\,q) = (1-x)^\beta x^\alpha, \quad 0 < x < 1,$$

so that $w(x;\alpha,\beta\,|\,q)$ is a $q$-analog of the beta density on [0, 1], and hence

$$\lim_{q\to 1} p_n(x;\alpha,\beta\,|\,q) = P_n^{(\alpha,\beta)}(x),$$

where $P_n^{(\alpha,\beta)}$ are the monic Jacobi polynomials on [0, 1]. Little $q$-Jacobi polynomials appear in representations of quantum SU(2) [9,10], and the special case of little $q$-Legendre polynomials was used to prove irrationality of a $q$-analog of the harmonic series and log 2 [14]. Their role in partitions was described in [1]. A detailed list of formulas for the little $q$-Jacobi polynomials can be found in [8, Section 3.12], but note that in that reference the polynomial $p_n(x;a,b\,|\,q)$ is not monic and that $a = q^\alpha$, $b = q^\beta$. Useful formulas are the *lowering operation*

$$\mathscr{D}_q p_n(x;\alpha,\beta\,|\,q) = \frac{1-q^n}{1-q}\,p_{n-1}(x;\alpha+1,\beta+1\,|\,q), \tag{1.5}$$

where $\mathscr{D}_q$ is the $q$-difference operator

$$\mathscr{D}_q f(x) = \begin{cases} \frac{f(x)-f(qx)}{(1-q)x} & \text{if } x \neq 0, \\ f'(0) & \text{if } x = 0 \end{cases} \tag{1.6}$$