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On Warnaar's elliptic matrix inversion and Karlsson–Minton-type elliptic hypergeometric series

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Abstract

Using Krattenthaler's operator method, we give a new proof of Warnaar's recent elliptic extension of Krattenthaler's matrix inversion. Further, using a theta function identity closely related to Warnaar's inversion, we derive summation and transformation formulas for elliptic hypergeometric series of Karlsson–Minton type. A special case yields a particular summation that was used by Warnaar to derive quadratic, cubic and quartic transformations for elliptic hypergeometric series. Starting from another theta function identity, we derive yet different summation and transformation formulas for elliptic hypergeometric series of Karlsson–Minton type. These latter identities seem quite unusual and appear to be new already in the trigonometric (i.e., $p = 0$) case.

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1. Introduction

Matrix inversions provide a fundamental tool for studying hypergeometric and basic hypergeometric (or q-) series. For instance, they underlie the celebrated Bailey transform [1]. For multiple hyper-

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geometric series, multidimensional matrix inversions have similarly proved to be a powerful tool, see [2,3,14–18,23–25].

Recently, a new class of generalized hypergeometric series was introduced, the elliptic hypergeometric series of Frenkel and Turaev [6]. In [31], Warnaar found an elliptic matrix inversion and used it to obtain several new quadratic, cubic and quartic summation and transformation formulas for elliptic hypergeometric series.

Warnaar's matrix inversion can be stated as follows [31, Lemma 3.2]. If

$$f_{nk} = \frac{\prod_{j=k}^{n-1} \theta(a_j c_k)\theta(a_j/c_k)}{\prod_{j=k+1}^{n} \theta(c_j c_k)\theta(c_j/c_k)} \tag{1.1a}$$

and

$$g_{kl} = \frac{c_l \theta(a_l c_l)\theta(a_l/c_l)}{c_k \theta(a_k c_k)\theta(a_k/c_k)} \frac{\prod_{j=l+1}^{k} \theta(a_j c_k)\theta(a_j/c_k)}{\prod_{j=l}^{k-1} \theta(c_j c_k)\theta(c_j/c_k)}, \tag{1.1b}$$

then the infinite lower-triangular matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are *inverses* of each other, i.e., the orthogonality relations

$$\sum_{k=l}^{n} f_{nk} g_{kl} = \delta_{nl}, \quad \text{for all } n, l \in \mathbb{Z} \tag{1.2}$$

and (equivalently)

$$\sum_{k=l}^{n} g_{nk} f_{kl} = \delta_{nl}, \quad \text{for all } n, l \in \mathbb{Z} \tag{1.3}$$

hold. In (1.1a) and (1.1b), $\theta(x)$ is the *theta function*, defined by

$$\theta(x) = \theta(x; p) := \prod_{j=0}^{\infty} (1 - xp^j)(1 - p^{j+1}/x),$$

for $|p| < 1$.

Note that $\theta(x)$ reduces to $1 - x$ for $p = 0$. In this case Warnaar's matrix inversion reduces to a result of Krattenthaler [13, Corollary], which in turn generalizes a large number of previously known explicit matrix inversions.

The present paper can be viewed as a spin-off of an attempt to obtain multivariable extensions of Warnaar's matrix inversion and use these to study elliptic hypergeometric series related to classical root systems. This led us to discover several aspects of Warnaar's result which are interesting already in the one-variable case. Multivariable extensions of these ideas are postponed to future publications.

Warnaar's proof of his inversion is based on Eq. (1.3), which is obtained as a special case of a more general identity, the latter being easily proved by induction. This approach seems difficult (though interesting) to generalize to the multivariable case. On the other hand, as was pointed out in [21], the identity (1.2) for Warnaar's inversion is equivalent to a partial fraction-type expansion for theta functions due to Gustafson, (2.2) below. This leads to a short proof of Warnaar's (and thus also Krattenthaler's) matrix inversion, which is described in Section 2.