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A relativistic hypergeometric function

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Abstract

We survey our work on a function generalizing ${}_2F_1$. This function is a joint eigenfunction of four Askey–Wilson-type hyperbolic difference operators, reducing to the Askey–Wilson polynomials for certain discrete values of the variables. It is defined by a contour integral generalizing the Barnes representation of ${}_2F_1$. It has various symmetries, including a hidden D_4 symmetry in the parameters. By means of the associated Hilbert space transform, the difference operators can be promoted to self-adjoint operators, provided the parameters vary over a certain polytope in the parameter space Π. For a dense subset of Π, parameter shifts give rise to an explicit evaluation in terms of rational functions of exponentials ('hyperbolic' functions and plane waves).

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1. Introduction

In the following, we review various papers concerned with a function $R(a_+, a_-, \mathbf{c}; v, \hat{v})$ generalizing the hypergeometric function ${}_2F_1(a, b, c; w)$, namely, Refs. [11,17,18] (referred to as I, II and III) and Ref. [20]. As is well known, the ${}_2F_1$-function can be used to diagonalize the nonrelativistic Schrödinger operator (2.12), which arises in the context of nonrelativistic Calogero–Moser systems. In [10] we

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introduced the R-function to diagonalize a generalization of (2.12) arising in the context of relativistic Calogero–Moser systems. The pertinent relativistic quantum operator amounts to an analytic difference operator (AΔO) of hyperbolic Askey–Wilson type.

Even though we do consider the nonrelativistic limit $R \to {}_2F_1$ in this survey, it is beyond our present scope to elaborate on the physical setting and Calogero–Moser context for the R-function. For information on these aspects we refer to our lecture notes [10]. The results obtained in I have been reviewed before from various complementary viewpoints in [12,14,15], viz., integrable systems, special functions, and sine-Gordon theory, resp. Accordingly, our account of results from I is terse and biased towards subjects that we need to sketch our more recent work in II, III and [20].

In the above-mentioned articles we have included a great many references to related work, pertinent to the context at issue. Since we are focusing on our results concerning the R-function (which, to our knowledge, has not been studied by other authors), we only mention here various papers where non-polynomial functions have been considered that are also solutions to an Askey–Wilson-type difference equation [7,3,5,24,8,6,22]. It is an open problem to make their relation to the R-function more explicit (cf. in this connection Section 6.6 in [14]).

We proceed to sketch the organization of this review. In Section 2, we recall some known lore on ${}_2F_1$, in a form that suits our later requirements. Section 3 has an auxiliary character, too. Here we collect some salient features of the hyperbolic gamma function from [9], which is the building block of the R-function.

This prepares the ground for Section 4, in which the R-function is defined. We also specify its analyticity properties and collect some manifest symmetries. In Section 5, we detail and discuss the most prominent feature of the R-function, namely its being a joint eigenfunction of four independent hyperbolic Askey–Wilson-type AΔOs.

Just as ${}_2F_1$ can be specialized to the Jacobi polynomials, the R-function can be specialized to the Askey–Wilson polynomials [1,4]. This is sketched in Section 6.

The results mentioned thus far date back to I. Section 7 is concerned with the main results obtained in II. As it turns out, the R-function has a hidden D_4 symmetry in the four coupling parameters $\mathbf{c} \in \mathbb{C}^4$. This symmetry is best understood in terms of a similarity transform $\mathscr{E}(a_+, a_-, \gamma; v, \hat{v})$, where γ is linear in a_+, a_- and $\mathbf{c}$, cf. (7.2). Indeed, the $\mathscr{E}$-function is D_4 invariant, cf. (7.16), whereas the R-function is only D_4 covariant. The $\mathscr{E}$-function also has plane wave asymptotics for Re $v \to \infty$, cf. (7.27)–(7.28).

In Section 8, we obtain the nonrelativistic limits of the R- and $\mathscr{E}$-functions and the four associated AΔOs, tying this in with the preparatory material in Section 2.

The Hilbert space eigenfunction transform corresponding to the $\mathscr{E}$-function is studied in III and surveyed in Sections 9 and 10. Section 9 concerns a sketch of our solution to the Plancherel problem (orthogonality and completeness). Along the way, the normalization integrals of the bound states arise in explicit form. For the ground state this gives rise to a hyperbolic analog of the (trigonometric) Askey–Wilson integral. Since this spin-off of our completeness proof is of considerable interest in itself, we have isolated it in Section 10. (See Stokman's preprint [23] for a quite different derivation of the relevant integral.)

A large amount of additional information can be obtained via an algebra of 32 parameter shifts. In particular, it can be shown that the R- and $\mathscr{E}$-functions have an elementary character (involving solely plane waves and hyperbolic functions) for a D_4 invariant dense set in the natural parameter space. We obtained these results in our recent paper [20] and review them in Section 11.