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Generalized associated polynomials and functions of second kind

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Abstract

Let $\{p_\nu\}_{\nu\in\mathbb{N}_0},\ \ p_\nu \in \Pi_\nu \backslash \Pi_{\nu-1}$, be a sequence of polynomials, generated by a three-term recurrence relation. Shifting the recurrence coefficients of the elements of $\{p_\nu\}_{\nu\in\mathbb{N}_0}$ we get a sequence of so-called associated polynomials, which play an important role in the theory of orthogonal polynomials. We generalize this concept of associating for arbitrary polynomials $v_n \in \Pi_n$. Especially, if v_n is expanded in terms of $p_\nu,\ \ \nu = 0, \ldots, n$, their associated polynomials are Clenshaw polynomials which are used in numerical mathematics. As a consequence of it we present some results from the viewpoint of associated polynomials and from the viewpoint of Clenshaw polynomials.

Analogously as for orthogonal polynomials we define functions of second kind for v_n. We prove some properties of them which depend on the generalized associated polynomials and functions of second kind for orthogonal polynomials.

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1. Introduction

Let $\{p_\nu\}_{\nu\geqslant 0}$ be a sequence of polynomials, satisfying the recurrence

$$\begin{aligned} &p_{-1}(x) = 0, \quad p_0(x) := 1, \\ &p_{n+2}(x) = (\alpha_{n+1}x - \beta_{n+1})p_{n+1}(x) - \gamma_{n+1}p_n(x), \qquad n \geqslant -1 \end{aligned} \tag{1.1}$$

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with $\alpha_n \neq 0$, $n \geqslant 0$. If $\gamma_n \neq 0$ for all $n \in \mathbb{N}_0$, then these polynomials are orthogonal with respect to a quasi-definite linear functional $\mathscr{L}$, defined on the space Π of all algebraic polynomials (cf. [2]). Furthermore, γ_0 can be chosen arbitrarily. For simplicity we choose $\gamma_0 := \mathscr{L}(p_0^2)$.

Shifting the recurrence coefficients in (1.1) we obtain a sequence $\{p_n^{(r)}\}_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, of polynomials, called r-associated polynomials, satisfying

$$\begin{aligned} &p_{-1}^{(r)}(x) = 0, \quad p_0^{(r)}(x) := 1, \\ &p_{n+2}^{(r)}(x) = (\alpha_{n+r+1}x - \beta_{n+r+1})p_{n+1}^{(r)}(x) - \gamma_{n+r+1}p_n^{(r)}(x), \quad n \geqslant -1. \end{aligned} \tag{1.2}$$

For $r = 0$ the superscripts can be omitted ($p_n^{(0)} = p_n$).

Besides (1.2), they satisfy a ‘dual’ recurrence relation [3,5,11]

$$\begin{aligned} &p_{-1}^{(k+2)} = 0, \quad p_0^{(k+1)} = 1, \\ &p_n^{(k)}(x) = (\alpha_k x - \beta_k)p_{n-1}^{(k+1)}(x) - \gamma_{k+1}p_{n-2}^{(k+2)}(x), \quad k \in \mathbb{N}_0, \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{1.3}$$

In this formula, polynomials with different polynomial degrees and degrees (order) of association are combined. Eqs. (1.1) and (1.3) are used to prove the ‘dual Christoffel–Darboux identity’

$$\begin{aligned} \sum_{k=0}^{n} \alpha_k p_k(x) p_{n-k}^{(k+1)}(y) &= \begin{cases} \frac{p_{n+1}(x)-p_{n+1}(y)}{x-y} =: [x, y; p_{n+1}], & x \neq y, \\ p'_{n+1}(x), & x = y, \end{cases} \\ &= \sum_{k=0}^{n} \alpha_k p_k(y) p_{n-k}^{(k+1)}(x). \end{aligned} \tag{1.4}$$

Using (1.4), it is easy to show that

$$p_{n-j}^{(j)}(x) = \frac{1}{\alpha_{j-1}\mathscr{L}(p_{j-1}^2)}(\mathscr{L}([\cdot, y; p_n]p_{j-1}))(x), \quad 1 \leqslant j \leqslant n. \tag{1.5}$$

Both formulas (1.4), (1.5) are proved in [11, Theorem 1]. Here, we made the assumption that $\mathscr{L}$ is quasi-definite, i.e. $\gamma_\nu \neq 0$ for all $\nu \in \mathbb{N}_0$, which is assumed to be true from now on. Because

$$\mathscr{L}(p_{m-1}^2) = \frac{\alpha_0}{\alpha_{m-1}} \prod_{\nu=0}^{m-1} \gamma_\nu, \tag{1.6}$$

we then have $\mathscr{L}(p_{m-1}^2) \neq 0$. Identity (1.5) gives us an idea how we can define a generalization of associated polynomials for arbitrary polynomials $v_n \in \Pi_n$ (Π_n denotes the space of all algebraic polynomials of degree $\leqslant n$). This will be done in the next Section 2 where some basic properties of these generalized associated polynomials are given, too. If v_n is expanded in terms of the p_ν, $\nu = 0, \ldots, n$, we show in Section 3 that these polynomials are Clenshaw polynomials which are used in various fields of numerical mathematics. Using this connection we present some applications of generalized associated polynomials. A special example is given in Section 4 where we consider divided differences and derivatives of v_n.

Generalized associated polynomials are useful in many mathematical areas. For example, Peherstorfer [6,7] used them (of order 1) to characterize positive quadrature formulas. In Section 3 we show how they can be used to evaluate polynomials and their derivatives with a Clenshaw-like algorithm (cf. [8]).