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Two linear transformations each tridiagonal with respect to an eigenbasis of the other: comments on the split decomposition

Paul Terwilliger

Mathematics Department, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706-1388, USA

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Abstract

Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both conditions below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call such a pair a *Leonard pair* on V. Referring to the above Leonard pair, it is known there exists a decomposition of V into a direct sum of one-dimensional subspaces, on which A acts in a lower bidiagonal fashion and A^* acts in an upper bidiagonal fashion. This is called the *split decomposition*. In this paper, we give two characterizations of a Leonard pair that involve the split decomposition.

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E-mail address: terwilli@math.wisc.edu (P. Terwilliger).

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1. Leonard pairs and Leonard systems

We begin by recalling the notion of a *Leonard pair* [6,12–18]. We will use the following terms. Let X denote a square matrix. Then X is called *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is called *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 (*Terwilliger [13, Definition 1.1]*). Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfies both (i) and (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2. According to a common notational convention A^* denotes the conjugate transpose of A. We are not using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

Our use of the name "Leonard pair" is motivated by a connection to a theorem of Leonard [2, p. 260]; [9], which involves the q-Racah polynomials [1]; [3, p. 162] and some related polynomials of the Askey scheme [7]. This connection is discussed in [13, Appendix A] and [15, Section 16]. See [4,5,8,10,19] for related topics.

In this paper, we obtain two characterizations of a Leonard pair. These characterizations are based on a concept which we call the *split decomposition*. We will formally define the split decomposition in Section 2, but roughly speaking, this is a decomposition of the underlying vector space into a direct sum of one-dimensional subspaces, with respect to which one element of the pair acts in a lower bidiagonal fashion and the other element of the pair acts in an upper bidiagonal fashion. In [13] we showed that every Leonard pair has a split decomposition. In the present paper, we consider a pair of linear transformations that is not necessarily a Leonard pair. We find a necessary and sufficient condition for this pair to have a split decomposition. Our main result along this line is Theorem 4.1. Now assuming the pair has a split decomposition, we give two necessary and sufficient conditions for this pair to be a Leonard pair. These conditions are stated in Theorems 5.1 and 5.2. These conditions are restated for a more concrete setting in Theorems 6.3 and 6.4.

When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract concept called a *Leonard system*. In order to define this we recall a few more terms. Let d denote a nonnegative integer. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices which have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. Let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of all $d+1$ by 1 matrices which have entries in $\mathbb{K}$. We index the rows by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$ under matrix multiplication. We observe this module is irreducible. For the rest of this paper we let $\mathscr{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. When we refer to an $\mathscr{A}$-module we mean a left $\mathscr{A}$-module. Let V denote an irreducible $\mathscr{A}$-module. We remark that