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Journal of Computational and Applied Mathematics 175 (2005) 183–194

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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On the stability of exponential fitting BDF algorithms

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Received 5 October 2003; received in revised form 5 April 2004

Abstract

We present BDF type formulas capable of the exact integration (with only round-off errors) of differential equations whose solutions are linear combinations of an exponential with parameter λ and ordinary polynomials. For $\lambda = 0$ new formulas reduces to the classical BDF formulas. Plots of their 0-stability regions in terms of λ are provided. Plots of their regions absolute stability that include all the negative real axis are provided. Numerical examples shows the efficiency of the proposed codes, specially when we are integrating stiff oscillatory problems.

Keywords: BDF methods; Exponential fitting; Stiff oscillatory problems

1. Introduction

Let us consider in the closed finite interval $[a, b]$ the IVP

$$y' = Ay + f(t, y), \tag{1}$$

where A is a $m \times m$ constant matrix whose eigenvalues have a negative real part, $y = y(t)$ is a real m-dimensional vector in a real variable t and $f(t, y)$ satisfies conditions for the existence and unicity of the solution. To solve this kind of problem many types of methods have been proposed in the search for good stability properties or higher orders. For example, the following explicit method:

$$\alpha_0 y_n = \alpha_0 e^{Ah} y_{n-1} + hf_{n-1}, \tag{2}$$

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2004.06.010

where $\alpha_0 = -Ah(1 - e^{Ah})^{-1}$, has appeared in the literature several times as example of a method of order one that integrates problem (1) when f is constant with no truncation error. This kind of multistep methods is widely known as exponential fitting or adapted algorithms (see [6] for a general theory).

When $A = 0$, this method reduces to the classical first-order BDF method. In this paper, we will try to extend the BDF formulas of two and three steps to methods with the mentioned properties. The methods presented here are developed in a fixed step mode, and could be extended to variable step mode. The only difficulty of these extensions will be the necessity of re-calculating the exponential function when the step size is changed; an efficient technique to avoid this computation is an open question.

For the classical BDF methods, the absolute stability regions of the explicit methods are poor, except for the first-order method, while the implicit methods work well up to sixth order. We will see that the same statement is true in our schemes although the regions of absolute stability of the implicit methods is clearly bigger. In fact the two and three-step explicit method are 0-unstable although they are exponential fitting.

The article is divided as follows: in Section 2, we will construct the methods using the generating function technique proposed by Vigo-Aguiar, in Section 3 we will study the 0-stability, in Section 4 we will provide a definition of absolute stability of the methods and some plots of those regions. Finally, in Section 5 we will compare our algorithms with well-known algorithms for the numerical solution of stiff problems special emphasis will be given to the case of stiff oscillatory problems.

2. Construction of fixed-step adapted-BDF methods

In Vigo-Aguiar [6,7] we find a general theory to construct the exponential fitting version of any multistep method. Based on this theory, we will produce methods of the BDF type that integrate (1) when $y(x)$ belongs to the space generated by the linear combinations of $\langle e^{Ax}, 1, x\rangle$ for the two-step method and $\langle e^{Ax}, 1, x, x^2\rangle$ for the three-step method. The parameter A times the step-length h, appearing in the coefficients of the method and we call it parameter of the method. It could be a matrix or a real number depending of the dimension of the problem.

In the case of low step methods, an alternative way to construct exponential fitting methods is to impose the exact integration of certain spaces of functions obtaining a system of equations whose solutions are the coefficient of the method.

Following any of the two ways of construction (and after some algebra), we have obtained the following expressions for the implicit methods ($s = 0$) and the explicit methods ($s = 1$):

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \alpha_3 y_{n-3} = h \quad f_{n-s}, \tag{3}$$

where the coefficients for the three-step method are

$$\begin{aligned} &\alpha_0 = \beta_0^s + \beta_1^s + \beta_2^s, \quad \alpha_1 = -(\beta_0^s + \beta_1^s + \beta_2^s)e^{Ah} - (\beta_1^s + 2\beta_2^s), \\ &\alpha_2 = (\beta_1^s + 2\beta_2^s)e^{Ah} + \beta_2^s, \quad \alpha_3 = -\beta_2^s e^{Ah}. \end{aligned} \tag{4}$$

For the two-step method the coefficients are the same but taking $\beta_2^s = 0$.