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Scattering theory for the Schrödinger equation with repulsive potential

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Abstract

We consider the scattering theory for the Schrödinger equation with $-\Delta - |x|^\alpha$ as a reference Hamiltonian, for $0 < \alpha \leqslant 2$, in any space dimension. We prove that, when this Hamiltonian is perturbed by a potential, the usual short range/long range condition is weakened: the limiting decay for the potential depends on the value of α, and is related to the growth of classical trajectories in the unperturbed case. The existence of wave operators and their asymptotic completeness are established thanks to Mourre estimates relying on new conjugate operators. We construct the asymptotic velocity and describe its spectrum. Some results are generalized to the case where $-|x|^\alpha$ is replaced by a general second order polynomial.

Résumé

Nous considérons la théorie de la diffusion pour l'équation de Schrödinger ayant $-\Delta - |x|^\alpha$ pour hamiltonien de référence, avec $0 < \alpha \leqslant 2$, en toute dimension d'espace. Nous démontrons que lorsque cet hamiltonien est perturbé par un potentiel, la notion habituelle de courte portée/longue

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portée est affaiblie : la décroissance limite de la perturbation dépend de la valeur de α, et est liée à la vitesse des trajectoires classiques dans le cas non perturbé. Nous établissons l'existence d'opérateurs d'ondes ainsi que leur complétude asymptotique grâce à des estimations de Mourre reposant sur de nouveaux opérateurs conjugués. En outre, nous construisons la vitesse asymptotique et nous décrivons son spectre. Enfin, nous généralisons certains résultats au cas où $-|x|^\alpha$ est remplacé par un polynôme du second degré.

MSC: 35B33; 35B40; 35P25; 47A40

Keywords: Scattering theory; Schrödinger equation; Harmonic potential; Mourre estimate

1. Introduction

The aim of this paper is to study the scattering theory for a large class of Hamiltonians with repulsive potential. We find optimal short range conditions for the perturbation, and prove asymptotic completeness under these conditions. The family of Hamiltonians is given by:

$$\mathsf{H}_{\alpha,0} = -\Delta - |x|^\alpha, \quad 0 < \alpha \leqslant 2; \qquad \mathsf{H}_\alpha = \mathsf{H}_{\alpha,0} + V_\alpha(x); \quad x \in \mathbb{R}^n,\ n \geqslant 1. \tag{1.1}$$

The main new feature with respect to the usual free Schrödinger operator $\mathsf{H}_{0,0} = -\Delta$ is the acceleration due to the potential $-|x|^\alpha$. The case $\alpha = 2$ is a borderline case: if $\alpha > 2$ classical trajectories reach infinite speed and $(\mathsf{H}_{\alpha,0}, C_0^\infty(\mathbb{R}^n))$ is not essentially self-adjoint (see [12]).

The consequence of the acceleration is that the usual position variable increases faster than t along the evolution. Roughly speaking, the usual short range condition is:

$$\left|V_0(x)\right| \lesssim \langle x\rangle^{-1-\varepsilon}, \tag{1.2}$$

for some $\varepsilon > 0$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. One expects it to be weakened in the case of H_α.

For the Stark Hamiltonian, associated to a constant electric field $E \in \mathbb{R}^n$ (see [8]), $-\Delta + E\cdot x$, it is well known that the short range condition (1.2) becomes $|V_s(x)| \lesssim \langle E\cdot x\rangle^{-1/2-\varepsilon}$. We refer to the papers by J.E. Avron and I.W. Herbst [2,18] for weaker conditions. The idea is that the drift caused by E (which may also model gravity, see, e.g., [33]) accelerates the particles in the direction of the electric field. This phenomenon has been observed for a larger class of Hamiltonians by M. Ben-Artzi [3,4]: generalizing the Stark Hamiltonian ($\alpha = 1$), let

$$\widehat{\mathsf{H}}_{\alpha,0} = -\Delta - \operatorname{sgn}(x_1)|x_1|^\alpha, \quad 0 < \alpha \leqslant 2; \qquad \widehat{\mathsf{H}}_\alpha = \widehat{\mathsf{H}}_{\alpha,0} + \widehat{V}_\alpha(x),$$

with $x = (x_1, x')$. In [4], asymptotic completeness is proved under the condition:

$$\left|\widehat{V}_\alpha\right| \lesssim M(x')\cdot\begin{cases}\langle x_1\rangle^{\alpha-\varepsilon} & \text{for } x_1 \leqslant 0,\\ \langle x_1\rangle^{-1+\alpha/2-\varepsilon} & \text{for } x_1 \geqslant 0,\end{cases} \tag{1.3}$$