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An efficient interpolation algorithm on anisotropic grids for functions with jump discontinuities in 2-D

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Abstract

In this paper we construct an algorithm that generates a sequence of continuous functions that approximate a given real valued function f of two variables that have jump discontinuities along a closed curve. The algorithm generates a sequence of triangulations of the domain of f. The triangulations include triangles with high aspect ratio along the curve where f has jumps. The sequence of functions generated by the algorithm are obtained by interpolating f on the triangulations using continuous piecewise polynomial functions. The approximation error of this algorithm is $\mathrm{O}(1/N^2)$ when the triangulation contains N triangles and when the error is measured in the L^1 norm. Algorithms that adaptively generate triangulations by local regular refinement produce approximation errors of size $\mathrm{O}(1/N)$, even if higher-order polynomial interpolation is used.

Keywords: Anisotropic triangulations; Adaptive mesh refinement; Polynomial interpolation; Local error estimate; Jump discontinuities

1. Introduction

In this work we consider the problem of approximating functions of two variables with jump discontinuities along a closed curve by means of continuous, piecewise polynomial functions on triangular grids. A model problem we will analyse consists in approximating a function f defined on a square Ω, with jump discontinuities along a closed curve Γ contained in Ω and such that f has value 1 in the region en-

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closed by Γ and 0 outside. The algorithm presented in this paper constructs a sequence of triangular grids $T_0, T_1, T_2, \ldots$ of Ω containing stretched triangles in the direction tangent to the curve Γ. For this model problem, we will show that the approximating functions f_j, where f_j is obtained by linear interpolation of f on T_j, satisfy $\|f - f_j\|_{L^1(\Omega)} = \mathrm{O}(1/N_j^2)$, being N_j the number of triangles on the grid T_j. For global quasi-uniform grids such error of approximation is $\mathrm{O}(1/\sqrt{N})$, and for local quasi-uniform grids the error is $\mathrm{O}(1/N)$, if N is the number of triangles on the grid. For our model problem these rates of convergence are the same if higher-order polynomial interpolation is used. This is because higher-order polynomial interpolation does not improve the order of convergence on regions with jumps, and because on regions where f is smooth f is constant and thus well approximated by linear polynomials. In the case where f is smooth on each region separated by Γ, but not necessarily constant, the above rates of convergence are retained if piecewise polynomial interpolation of degree 3 is used instead of 1; higher-order interpolation does not improve the rates. Our algorithm is based on an anisotropic refinement to obtain a hierarchical sequence of grids, where triangles with very small or large interior angles are part of the grid. This is opposed to the regular refinement algorithms to obtain local quasi-uniform triangulations, where new grids are obtained by dividing some triangles into 4 triangles of equal shape to the original one, and where the aspect ratio of all the triangles in all the grids is uniformly bounded.

This paper is organized as follows: Section 2 describes the model problem. Section 3 is about regular and anisotropic grid refinements. The notation and an algorithm for anisotropic refinement is discussed in Sections 4 and 5. In Section 6 some numerical tests of the anisotropic refinement algorithm are presented. Section 7 studies more general cases than our model problem. Proofs of lemmas and theorems are provided in Appendix A.

2. The model problem

As a model problem to show the performance of our algorithm to approximate discontinuous functions f, we took as f the indicator function on regions contained on a square: let Γ be a closed and smooth simple curve in the plane, and let Ω be a square that contains Γ. Let D be the closure of the region in the plane bounded by Γ. Define $f:\Omega \to \mathbb{R}$ as

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \in D, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Given a triangular grid T of Ω (see Remark 2.1), denote by f_T the continuous and piecewise linear interpolant of f. Thus, f_T is the continuous function that satisfies:

- f_T is affine-linear in each triangle $K \in T$
- $$f_T(\mathbf{a}) = f(\mathbf{a}), \tag{2}$$

  for all vertices $\mathbf{a}$ of all triangles $K \in T$.

The interpolant function f_T approximates f with an error given in the L^1 norm by:

$$\|f - f_T\|_{L^1(\Omega)} = \int_\Omega |f - f_T| \,\mathrm{d}x\,\mathrm{d}y. \tag{3}$$