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An iterated pseudospectral method for delay partial differential equations

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Abstract

The Chebyshev pseudospectral semi-discretization preconditioned by a transformation in space is applied to delay partial differential equations. The Jacobi waveform relaxation method is then applied to the resulting semi-discrete delay systems, which gives simple systems of ordinary equations $\frac{\mathrm{d}}{\mathrm{d}t}U^k(t) = M_\alpha U^k(t) + f_\alpha(t, U_t^{k-1})$. Here, M_α is a diagonal matrix, which depends on a parameter $\alpha \in [0, 1]$, which is used in the transformation in space, k is the index of waveform relaxation iterations, U_t^k is a functional argument computed from the previous iterate and the function f_α, like the matrix M_α, depends on the process of semi-discretization. Jacobi waveform relaxation splitting has the advantage of straightforward (because M_α is diagonal) application of implicit numerical methods for time integration. Another advantage of Jacobi waveform relaxation is that the resulting systems of ordinary differential equations can be efficiently integrated in a parallel computing environment. The spatial transformation is used to speed up the convergence of waveform relaxation by preconditioning the Chebyshev pseudospectral differentiation matrix. We study the relationship between the parameter α and the convergence of waveform relaxation with error bounds derived here for the iteration process. We find that convergence of waveform relaxation improves as α increases, with the greatest improvement at $\alpha = 1$. These results are confirmed by numerical experiments for hyperbolic, parabolic and mixed hyperbolic-parabolic problems with and without delay terms.

Keywords: Hyperbolic; Parabolic; Delay equations; Chebyshev pseudospectral method; Kosloff Tal-Ezer transformation; Waveform relaxation; Error estimations

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1. Introduction

1.1. Delay partial differential problems

In this paper we study numerical solutions to the non-homogeneous initial boundary value problem with functional term

$$\frac{\partial}{\partial t}u(x,t) = \varepsilon\frac{\partial^2}{\partial x^2}u(x,t) + c\frac{\partial}{\partial x}u(x,t) + g(x,t,u_{(x,t)}), \quad -L \leqslant x \leqslant L,\ 0 < t \leqslant T,$$
$$u(x,t) = f_0(x,t), \quad -\tau_0 \leqslant t \leqslant 0,\ -L \leqslant x \leqslant L. \tag{1.1}$$

Here, $\varepsilon \geqslant 0$, $c \in \mathbb{R}$, $\tau_0 \geqslant 0$, $L > 0$ and $T > 0$ are given constants. Choices for c and ε have vastly different behaviour, i.e., $\varepsilon = 0$ gives the hyperbolic one-way wave equation, $c = 0$ gives the parabolic heat equation, while both $\varepsilon \neq 0$ and $c \neq 0$ gives the parabolic advection–diffusion equation. Different types of boundary conditions are required for the two cases $\varepsilon \neq 0$ and $\varepsilon = 0$. For the parabolic case ($\varepsilon \neq 0$) there are two boundary conditions

$$u(\pm L, T) = f_\pm(t), \tag{1.2}$$

while for the hyperbolic case ($\varepsilon = 0$, $c \neq 0$) there is one boundary condition, either

$$u(L,t) = f_+(t) \quad (\text{if } c > 0) \quad \text{or} \quad u(-L,t) = f_-(t) \quad (\text{if } c < 0). \tag{1.3}$$

Here, f_0 and $f_\pm$ are given initial and boundary functions while the function $u_{(x,t)}$ for $(x,t) \in [-L,L] \times [0,T]$ is defined by

$$u_{(x,t)}(\tau) = u(x, t+\tau), \quad \tau \in [-\tau_0, 0], \tag{1.4}$$

and $g : [-L,L] \times [0,T] \times C([-\tau_0,0],\mathbb{R}) \to \mathbb{R}$ is a continuous function. Eq. (1.1) includes, for example, integro-differential equations

$$\frac{\partial}{\partial t}u(x,t) = \varepsilon\frac{\partial^2}{\partial x^2}u(x,t) + c\frac{\partial}{\partial x}u(x,t) + G\left(x, t, \int_{-\tau_0}^{0} u(x,t+\tau)\,\mathrm{d}\tau\right), \tag{1.5}$$

and delay equations

$$\frac{\partial}{\partial t}u(x,t) = \varepsilon\frac{\partial^2}{\partial x^2}u(x,t) + c\frac{\partial}{\partial x}u(x,t) + G\big(x, t, u(x, t-\tau_0)\big), \tag{1.6}$$

cp. [28, Section 3]. Here, $G : [-L,L] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. If G is given in (1.5) or (1.6) then the function g is

$$g(x,t,v) = G\left(x, t, \int_{-\tau_0}^{0} v(\tau)\,\mathrm{d}\tau\right)$$

or

$$g(x,t,v) = G\big(x, t, v(-\tau_0)\big),$$

respectively with $v \in C([-\tau_0,0],\mathbb{R})$. Functional problems like (1.1) are used to model cancer cells in human tumors, see [3]. For other applications in population dynamics we refer the reader to [9].