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A moving mesh finite element algorithm for the adaptive solution of time-dependent partial differential equations with moving boundaries

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Abstract

A moving mesh finite element algorithm is proposed for the adaptive solution of nonlinear diffusion equations with moving boundaries in one and two dimensions. The moving mesh equations are based upon conserving a local proportion, within each patch of finite elements, of the total "mass" that is present in the projected initial data. The accuracy of the algorithm is carefully assessed through quantitative comparison with known similarity solutions, and its robustness is tested on more general problems.

Applications are shown to a variety of problems involving time-dependent partial differential equations with moving boundaries. Problems which conserve mass, such as the porous medium equation and a fourth order nonlinear diffusion problem, can be treated by a simplified form of the method, while problems which do not conserve mass require the full theory.

Keywords: Time-dependent nonlinear diffusion; Moving boundaries; Finite element method; Lagrangian meshes; Conservation of mass

1. Introduction

In this paper an adaptive finite element method is proposed for the solution of partial differential equations (PDEs) with moving boundaries, using a moving mesh. The approach is prompted by recent

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interest in geometric integration and scale invariance (see for example [10] and references therein) which has rekindled interest in the use of adaptive moving meshes. Scale invariance treats independent and dependent variables alike, suggesting that both solution and mesh should be varied when designing numerical schemes to inherit this property.

The use of moving meshes has been proposed in many different contexts over the past three decades, ranging from phase change problems [4,8,21], blow-up problems [9] or hyperbolic conservation laws [16], to general classes of time-dependent problem [2,15,22]. It is apparent from this significant body of research that moving grids have much to offer in terms of improved efficiency, although, in order to provide robust and reliable software, they must often be applied in conjunction with other adaptive techniques such as local remeshing [24] or order enrichment [14]. In addition, when there is sufficient a priori knowledge to ensure that the initial mesh provides an adequate resolution to avoid the need for later refinement, moving grids can provide an extremely powerful computational tool in their own right.

In the approach taken in this work the moving mesh equations are based upon conserving the local proportion, within each patch of finite elements, of the total integral (mass) of the dependent variable across the domain. Although not considered here, the integral may be generalised to conserve other quantities [6,7], yielding an approach similar to that of using a monitor function to control the movement of the mesh, as in the moving mesh partial differential equation (MMPDE) method [4,17] for example. It is also strongly related to the deformation method of Liao and co-workers [19,20] and to the geometric conservation law (GCL) method of Cao, Huang and Russell [11] (see also [25]). In particular the idea of a mesh velocity potential proposed in [11] is exploited in order to obtain uniqueness of the mesh velocity in greater than one space dimension.

The organisation of the paper is as follows. In the next section strong and weak formulations of the PDE in a moving reference frame are discussed, together with the local conservation principle upon which the proposed method is based. Section 3 contains the details of the method, which is a generalisation of the one-dimensional finite volume approach described in [6,7]. Applications are presented in Sections 4 and 5. The mass conserving applications in Section 4, for which the theory simplifies, consist of moving boundary problems governed by the porous medium equation (PME) and a fourth order nonlinear diffusion problem. Numerical results are presented in this section for comparison against known similarity solutions (in order to assess the accuracy of the technique), as well as problems for which there is no known analytic solution (to assess its robustness). Section 5 is devoted to the application of the technique to two non-mass-conserving problems, consisting of a one-phase Stefan problem and a diffusion problem with a negative source term. The paper concludes with a discussion of the work and a number of suggestions as to how it may be extended.

2. Background

Throughout this and the following section an abstract time-dependent partial differential equation (PDE) of the general form

$$\frac{\partial u}{\partial t} = Lu, \tag{1}$$

on a time-dependent domain $\Omega(t)$ will be considered. Specific equations will be treated in the subsequent sections. In the PDE (1) $u = u(\mathbf{x}, t)$ is defined in a fixed frame of reference with coordinate $\mathbf{x}$ at time t and