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An adaptive stabilized finite element scheme for the advection–reaction–diffusion equation

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Abstract

An adaptive finite element scheme for the advection–reaction–diffusion equation is introduced and analyzed. This scheme is based on a stabilized finite element method combined with a residual error estimator. The estimator is proved to be reliable and efficient. More precisely, global upper and local lower error estimates with constants depending at most on the local mesh Peclet number are proved. The effectiveness of this approach is illustrated by several numerical experiments.

Keywords: Advection–reaction–diffusion problem; Boundary and inner layers; A posteriori error estimates; Stabilized finite elements

1. Introduction

This paper deals with the advection–diffusion–reaction equation. This kind of problems arise in many applications, for instance, when linearizing the Navier–Stokes problem, to model pollutant transport and degradation in aquatic media, etc. In particular, our work is motivated by the need of an efficient scheme to be used in a water quality model for the Bío Bío River in Chile.

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Especially interesting is the case when advective or reactive terms are dominant. In this case, the solution of the equation frequently has exponential or parabolic boundary layers (for details see [9]). The standard Galerkin approximation usually fails in this situation because this method introduces nonphysical oscillations. A possible remedy is to add to the variational formulation some numerical diffusion terms to stabilize the finite element solution. Some examples of this approach are the streamline upwind Petrov–Galerkin method (SUPG) (see [3]), the Galerkin least squares approximation (GLS) (see [6]), the Douglas–Wang method (see [5]), the 'unusual' stabilized finite element method (USFEM) (see [7]), and the residual-free bubbles approximation (RFB) (see [2]). The drawback with most of these methods is that the solution layers are not very well resolved, because of the numerical diffusion added to the discretization.

In spite of the abundent literature on adaptivity (see, for instance [14]), there are not so many references dealing with *a posteriori* techniques for this equation. The reason for this is that most of the standard error estimators involve equivalence constants depending on negative powers of the diffusion parameter, which lead to very poor results in the advective or reactive dominated cases. An error estimator which is robust in the sense of leading to global upper and local lower bounds depending at most on the local mesh Peclet number has been developed by Verfürth (see [12,13]). Using these results Sangalli has analyzed a residual *a posteriori* error estimate for the residual-free bubbles scheme (see [10]). On the other hand, Knopp et al. have developed some *a posteriori* error estimates using a stabilized scheme combined with a shock-capturing technique to control the local oscillations in the crosswind direction (see [8]). Finally, Wang has introduced an error estimate for the advection–diffusion equation based on the solution of local problems on each element of the triangulation (see [15]).

In this paper we introduce and analyze from theoretical and experimental points of view an adaptive scheme to efficiently solve the advection–reaction–diffusion equation. This scheme is based on the stabilized finite element method introduced in [7] combined with an error estimator similar to the one developed in [13]. We prove global upper and local lower error estimates in the energy norm, with constants which only depend on the shape-regularity of the mesh, the polynomial degree of the finite element approximating space and the local mesh Peclet number. We perform several numerical experiments to show the effectiveness of our approach to capture boundary and inner layers very sharply and without significant oscillations. The experiments also show that the scheme attains optimal order of convergence.

The paper is organized as follows. In Section 2 we recall the advection–diffusion–reaction problem under consideration and the stabilized scheme. In Section 3 we define an *a posteriori* error estimator and prove its equivalence with the energy norm of the finite element approximation error. Finally, in Section 4, we introduce the adaptive scheme and report the results of the numerical tests.

2. A stabilized method for a model problem

Our model problem is the advection–reaction–diffusion equation

$$\begin{cases} -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u + bu = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_{\mathrm{D}}, \\ \varepsilon\dfrac{\partial u}{\partial \boldsymbol{n}} = g \quad \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with a Lipschitz boundary $\Gamma = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$, with $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$. We denote by $\boldsymbol{n}$ the outer unit normal vector to Γ.