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A numerical solver for the primitive equations of the ocean using term-by-term stabilization ☆

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Abstract

In this work we introduce and analyze a numerical approximation of the primitive equations of the ocean by means of stabilized finite elements. We use a reduced formulation of these equations which only includes the (3D) horizontal velocity and the (2D) surface pressure. This, combined with the use of stabilized finite elements, provides a large reduction of degrees of freedom in comparison with previous mixed methods. The use of isoparametric prismatic finite elements provides good geometric adaptability to the topography. We perform an analysis of stability and convergence using the concept of static condensation on bubble spaces. Finally, we test our stabilized approximations in flows with complex 3D structure, including a real-life application. Specifically, we simulate the wind-driven circulation in Lake Neuchâtel.

Keywords: Oceanography; Primitive equations; Mixed finite elements; Stabilized finite elements

1. Introduction

This work introduces a finite element solver with reduced computational complexity for the primitive equations of the ocean.

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Although two-dimensional approaches of oceanic flows can work remarkably well if the density is approximately constant (cf. Ambrosi et al. [1]), Oceanic flows present an essentially three-dimensional nature. This is caused by the asymmetry of the coasts and bottom, the Coriolis force and different up/down-welling mechanisms. The upwelling phenomenon which is coupled with Ekman and baroclinic effects, is of particular relevance, for it is at the basis of the development of life in the seas.

In this work we deal with the numerical solution of a quasi-3D model: the primitive equations of the ocean (cf. Lions et al. [33]). The primitive equations govern the behavior of oceanic flows for large time and horizontal spatial scales. Their steady version are found in the literature as a model for the long-time equilibrium flow obtained from certain initial conditions of interest (Speich et al. [40], Zuur and Dietrich [43]). This is sometimes called *secular flow*. In other instances (Deleersnijder [24], Espino Infantes [25]), the stationary solution is understood as the component of the flow induced by a characteristic wind or inflow in the area. In this case, the solution is sometimes referred to as *background flow*.

The numerical solvers of the primitive equations have been set up since about 1970. A pioneer work in this sense is Bryan [14]. Since then, several numerical methods have been used, mainly finite differences schemes (for example, Casulli and Cattani [16], Casulli and Cheng [17] and Zuur [42]). This and other methods are in continuous development (see Kowalik and Murty [28] or Vreugdenhil [41] for a review). In particular, a number of finite element techniques have been introduced to solve 3D geophysical flows over the last decade. See, for instance, Bates and Hervouet [6], Besson [8], Miglio et al. [35] and Laydi [31].

The numerical solution of the primitive equations of the ocean faces hard difficulties. On one hand, it requires the use of very large amounts of degrees of freedom, specially when dealing with large areas of complex geometry (irregular boundaries and large bathymetry gradients). The irregularity of the bottom is usually treated by finite difference and spectral methods by means of a change of variables, called σ-transformation (generally ascribed to Phillips [38]). By doing so, the computational domain is turned into a cylinder. However, the transformation of horizontal derivatives renders more complex the equations. Moreover, it can only be applied if the water depth is uniformly positive. As usually this is not the case, the domain is approximated by a subdomain limited by vertical walls (the *talus*) that replace the shore.

On the other hand, there exist two stability restrictions which must be conveniently treated by any numerical discretization. One of them is imposed by the incompressibility of the flow. In the context of numerical approximations by finite elements, it is usually overcome by introducing additional stabilizing degrees of freedom to the discrete velocities space in order to have a discrete inf–sup condition between the velocity-pressure spaces to be satisfied (cf. Azérad [3]). This largely increases the number of unknowns in the discretization.

The other stability restriction is due to convection dominance. This leads to the appearance of spurious oscillations in the numerical solution obtained by the Galerkin method. These oscillations disappear for small enough grid sizes, leading to additional increases of computational complexity.

Our strategy to overcome the mentioned difficulties in this work is three-fold:

- We use a mathematical formulation of the primitive equations of the ocean that include a minimal number of unknowns. This formulation only involves the surface pressure and the horizontal component of the velocity.
- The difficulty associated with rough bottoms is circumvented with the use of isoparametric prismatic finite elements following the contours of the topography. These contours correspond to the level surfaces of the classical σ-coordinate. There is no need to apply the σ-transformation to the differential