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Nonlinear operator integration factor splitting for the shallow water equations

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Abstract

The purpose of this paper is to explore an alternative to the traditional interpolation based semi-Lagrangian time integrators employed in atmospheric models. A novel aspect of the present study is that operator splitting is applied to a purely hyperbolic problem rather than the incompressible Navier–Stokes equations. The underlying theory of operator integration factor splitting is reviewed and the equivalence with semi-Lagrangian schemes is established. A nonlinear variant of integration factor splitting is proposed where the advection operator is expressed in terms of the relative vorticity and kinetic energy. To preserve stability, a fourth order Runge–Kutta scheme is applied for sub-stepping. An analysis of splitting errors reveals that OIFS is compatible with the order conditions for linear multi-step methods. The new scheme is implemented in a spectral element shallow water model using an implicit second order backward differentiation formula for Coriolis and gravity wave terms. Numerical results for standard test problems demonstrate that much larger time steps are possible.

Keywords: Integration factor; Multistep methods; High-order; Spectral elements

1. Introduction

The seminal work of Robert [25] led to a six-fold increase over the explicit time step for atmospheric general circulation models. To achieve such dramatic gains without recourse to a fully implicit integrator, a semi-Lagrangian treatment of advection was combined with a semi-implicit scheme for the stiff terms responsible for gravity waves. Initially, semi-implicit semi-Lagrangian time-stepping was applied to hy-

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perbolic problems, discretized using low-order finite-differences and finite elements [30]. However, the method was soon extended to global models based on the spectral transform [24]. The traditional semi-Lagrangian algorithm implemented in atmospheric models relies on backward trajectory integration and upstream interpolation. In effect, the numerical domain of dependence is shifted to an upstream grid cell and, for advective CFL numbers $C < 1$ with linear interpolation, is equivalent to upwind finite differencing. Upwind schemes are known to be diffusive and thus cubic interpolation has been generally adopted [29]. Indeed, McCalpin [20] has shown that high-degree polynomials are required in order to mitigate the inherent numerical dissipation and dispersion errors associated with semi-Lagrangian advection.

Bartello and Thomas [2] analyzed the cost-effectiveness of the backward semi-Lagrangian scheme in the context of geophysical flows. Their analysis was restricted to atmospheric models using low-order finite differences and either 2D or 3D Lagrangian interpolants. In the enstrophy cascade of homogeneous quasi-geostrophic turbulence, the authors concluded that high efficiency gains are possible over low-order Eulerian integrators. However, the gains are at best marginal in the case of a 3D Kolmogorov energy cascade. The cascade interpolation procedures of Purser and Leslie [22] and Nair et al. [21] reduce the computational complexity from $\mathcal{O}(N^d)$ to $\mathcal{O}(N)$ per grid point, where N is the degree of the interpolating polynomial in d space dimensions. The order of accuracy of these methods has not been formally established. Nevertheless, the time scale separation between Lagrangian and Eulerian frames is more restrictive for small-scale atmospheric dynamics when $E(k) \sim k^{-5/3}$ and the semi-Lagrangian scheme is only cost-effective at very high spatial resolutions.

The combination of the semi-Lagrangian approach together with a high-order spectral element space discretization, applied to the advection–diffusion equation, is described in Giraldo [13]. An important result of this study is that numerical dissipation and dispersion errors for the combined scheme are completely eliminated for polynomial order $N \geqslant 4$. More recently, Giraldo et al. [14] reported numerical results for a semi-Lagrangian semi-implicit shallow water model. Extension of the scheme to the hydrostatic primitive equations is discussed in Giraldo and Rosmond [15]. Motivated by the efficiency gains for advection–diffusion, Xiu and Karniadakis [34] applied a semi-Lagrangian spectral element (SESL) method to the incompressible Navier–Stokes equations. For laminar and transitional flows, they observed efficiency gains ranging from four to ten times over an Eulerian SE scheme. Xu et al. [36] simulated turbulent channel flow using a mixed spectral discretization. A ten-fold increase in the time-step was obtained, but only at the break-even point in computational efficiency due to the use of global interpolants. These results also confirm the error analysis of Falcone and Ferretti [10], who show that the overall error is *not* monotonic and can actually decrease as the time step increases for a particular choice of the spatial resolution.

In high-order methods, the computational cost of upstream interpolation for an Nth order discretization in d space dimensions is $\mathcal{O}(N^d)$ per degree of freedom. Given a spectral element discretization consisting of K elements of order N, there are KN^d grid points and the total interpolation cost is $\mathcal{O}(KN^{2d})$. By comparison, the cost of Eulerian operator evaluations scales as $\mathcal{O}(KN^{d+1})$. For example, advection of a scalar requires dKN^{d+1} operations [11]. A potentially lower cost alternative to interpolation is the operator integrating factor splitting (OIFS) method of Maday et al. [19] which relies on Eulerian sub-stepping of the advection equation. If the total number of sub-steps per time step is less than N^{d-1}, then OIFS should be more efficient. Boyd [4] observed that both semi-Lagrangian and OIFS algorithms are members of a broader class of integration factor methods.

There are several motivations for our evaluation of integration factor methods in the context of spectral elements applied to geophysical flows. The advective CFL number scales as $\mathcal{O}(N^{-2})$ and is more