Available online at www.sciencedirect.com

SCIENCE DIRECT®

Applied Numerical Mathematics 55 (2005) 403–424

APPLIED NUMERICAL MATHEMATICS

www.elsevier.com/locate/apnum

Reduced-basis methods for elliptic equations in sub-domains with a posteriori error bounds and adaptivity ☆

Gianluigi Rozza

Modelling and Scientific Computing (CMCS), École Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland

Available online 21 December 2004

Abstract

We present an application in multi-parametrized sub-domains based on a technique for the rapid and reliable prediction of linear-functional outputs of elliptic coercive partial differential equations with affine parameter dependence (reduced-basis methods). The main components are (i) rapidly convergent global reduced-basis approximations—Galerkin projection onto a space W_N spanned by solutions of the governing equation at N selected points in parameter space (chosen by an adaptive procedure to minimize the estimated error and the *effectivity*; (ii) *a posteriori* error estimation—relaxations of the error-residual equation that provide inexpensive bounds for the error in the outputs of interest; and (iii) off-line/on-line computational procedures—methods which decouple the generation and projection stages of the approximation process. The operation count for the on-line stage—in which, given a new parameter value, we calculate the output of interest and associated error bound—depends only on N (typically very small) and the parametric complexity of the problem; the method is thus ideally suited for the repeated and rapid evaluations required in the context of parameter estimation, design, optimization, and real-time control. The application is based on a heat transfer problem in a parametrized geometry in view of haemodynamics applications and biomechanical devices optimization, such as the bypass configuration problem.

Keywords: Elliptic parametrized partial differential equations; Reduced-basis methods; Output bounds; Galerkin approximation; A posteriori error estimation; Adaptive approximation procedure

☆ This work has been developed in collaboration with Prof. A.T. Patera and Dr. K. Veroy, during a visiting period of the author Massachusetts Institute of Technology, Mechanical Eng. Department, Fluids Laboratory, Cambridge, MA, USA.

E-mail address: gianluigi.rozza@epfl.ch (G. Rozza).

0168-9274/$30.00 © 2004 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.apnum.2004.11.004

1. Introduction

The optimization, control, design and characterization of an engineering component or system requires the prediction of certain "quantities of interest", or performance metrics, which we shall denote *outputs*—for example, velocity field, maximum stresses, maximum temperatures, heat transfer rates, flow rates, vorticity, or lifts and drags. These outputs are typically expressed as functionals of field variables associated with a parametrized partial differential equation which describes the physical behavior of the component or system. The parameters, which we shall denote *inputs*, serve to identify a particular "configuration" of the components: these inputs may represent design or decision variables, such as geometry, or characterization variables, such as physical properties—for example, in inverse design problems. We thus get an implicit *input–output* relationship, evaluation of which demands solution of the underlying partial differential equations. See [22] for a detailed presentation of design problems and some examples.

The development of computational methods is permitting *rapid* and *reliable* evaluation of this partial-differential-equation-induced input–output relationship in the design, optimization and control contexts. See recent developments in [17]. The approach used is based on the reduced-basis method, first introduced in the late 1970s for nonlinear structural analysis, and later developed more broadly in the 1980s and 1990s [1–3,8,9,16]. The reduced-basis method recognizes that the field variable is not, in fact, some arbitrary member of the infinite-dimensional solution space associated with the partial differential equation; rather, the field variable resides, or evolves, on a much lower-dimensional manifold induced by the parametric dependence.

In the application we use *global* approximation spaces; second, we make rigorous *a posteriori error estimations*; and third, we exploit *off-line*/*on-line* computational decompositions (see [1] for application of this strategy within the reduced-basis context). These three steps allow us—for the restricted but important class of "parameter-affine" problems—to reliably decouple the generation and projection stages of reduced-basis approximation, thereby effecting computational economies of several orders of magnitude.

In Section 2 we present the problem statement. In Section 3 we describe the *a posteriori* error estimation framework. In Section 4 we recall the *a priori* convergence theory applied also to our output bounds and not only to approximate solution. In Section 5 we study our procedure to control N more tightly and to apply our error bound adaptively in the choice of μ parameters family. Finally, in Section 6, we present the application and in Section 7 the numerical results for our "model-problem" example. In Section 8, we introduce possible developments for viscous flow and shape optimization [7] applied, for example, to haemodynamics [14] within the reduced-basis framework.

2. Problem formulation

We first introduce a Hilbert space Y, and an associated inner product and a norm, $(\cdot,\cdot)$ and $\|\cdot\| \equiv (\cdot,\cdot)^{1/2}$, respectively. We next introduce the dual space of Y, Y', and the associated duality pairing between Y and Y', ${}_{Y'}\langle\cdot,\cdot\rangle_Y \equiv \langle\cdot,\cdot\rangle$.