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Orthogonal Laurent polynomials and quadrature formulas for unbounded intervals: II. Interpolatory rules

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Abstract

We study the convergence of quadrature formulas for integrals over the positive real line with an arbitrary (possibly complex) distribution function. The nodes of the quadrature formulas are the zeros of orthogonal Laurent polynomials with respect to an auxiliary distribution function and a certain nesting. The quadratures are called interpolatory (product) formulas. The class of functions for which convergence holds is characterized in terms of the moments of the auxiliary distribution function. We also include the convergence analysis of related two-point Padé-type approximants to the Stieltjes transform of the given distribution function. Finally, some illustrative numerical examples are also given.

Keywords: Laurent polynomials; Gaussian quadrature; Interpolatory quadrature; Error estimates

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1. Introduction

In this paper we shall be mainly concerned with the estimation of integrals

$$I(f,\mu)=\int_a^b f(x)\mu(x)\,\mathrm{d}x,\quad 0\leqslant a<b\leqslant+\infty, \tag{1.1}$$

with μ L_1-Lebesgue-integrable and f at least Riemann-integrable whose singularities can only be the origin and/or infinity.

To approximate $I(f,\mu)$ we use quadrature rules of the form

$$I_n(f,\mu)=\sum_{j=1}^n A_{jn}f(x_{jn}) \tag{1.2}$$

(which are called *product integration rules*—see [22]), whose nodes $\{x_{jn}\}_1^n$ are preassigned in (a,b) and the weights or coefficients $\{A_{jn}\}_1^n$ are determined by requiring that (1.2) is exact for functions f in a linear space of dimension at least n. Thus, when the moment integrals

$$c_k=\int_a^b x^k\mu(x)\,\mathrm{d}x,\quad k=0,1,\ldots,$$

exist and are easily computed, then the weights are defined by imposing that

$$I_n(P,\mu)=I(P,\mu),\quad \forall P\in\Pi_{n-1}, \tag{1.3}$$

or equivalently by requiring $I_n(x^k,\mu)=I(x^k,\mu)=c_k$, $k=0,1,\ldots,n-1$ (we use the notation Π_k to denote the space of polynomials of degree at most k for k a nonnegative integer, while Π will denote the space of all polynomials).

In fact, μ could also be a weight function on (a,b), i.e., $\mu(x)>0$ a.e. on (a,b). In this case, by an appropriate choice of the nodes $\{x_{jn}\}$, formulas of the form (1.2) can be found that integrate exactly all polynomials up to a degree that is much higher than $n-1$. When the formulas have the highest possible degree of exactness in the set of polynomials that can be obtained in this way, they are called *Gaussian formulas*. However if μ is not a "standard" weight function (see, e.g., [13]), the calculation of the Gaussian formulas requires a long computational process, with possible numerical instability problems. For this reason, formula (1.2) satisfying (1.3) with "easily computable nodes" could be desirable even when μ is a weight function. Such quadratures should exhibit some nice properties so that their accuracy and efficiency can be assured. Concerning the latter properties, we define the following. A sequence of rules $\{I_n(f,\mu)\}_1^\infty$ like (1.2) is said to be *convergent in a class* $\mathcal{A}$ iff $\lim_{n\to\infty} I_n(f,\mu)=I(f,\mu)$ for all $f\in\mathcal{A}$. Obviously, it seems natural to make the class $\mathcal{A}$ as large as possible. On the other hand, a sequence $\{I_n(f,\mu)\}_1^\infty$ is said to be (numerically) *stable* if there exists a positive constant M (independent of n) such that

$$\sum_{j=1}^n |A_{jn}|\leqslant M,\quad n=1,2,\ldots. \tag{1.4}$$

Condition (1.4) means that the possible roundoff errors in the evaluation of $f(x_{jn})$ remain under control during the computation. On the other hand, it should be also noticed that the success of the rules (1.2)