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Adaptive radial basis function methods for time dependent partial differential equations

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Abstract

Radial basis function (RBF) methods have shown the potential to be a universal grid free method for the numerical solution of partial differential equations. Both global and compactly supported basis functions may be used in the methods to achieve a higher order of accuracy. In this paper, we take advantage of the grid free property of the methods and use an adaptive algorithm to choose the location of the collocation points. The RBF methods produce results similar to the more well-known and analyzed spectral methods, but while allowing greater flexibility in the choice of grid point locations. The adaptive RBF methods are most successful when the basis functions are chosen so that the PDE solution can be approximated well with a small number of the basis functions.

Keywords: Radial basis function; Partial differential equations; Adaptive grid

1. Introduction

Ultimately, we are interested in adaptive radial basis function (RBF) PDE algorithms in two and three spatial dimensions. In this paper, we gain insight in one dimension before proceeding to higher dimensions. The implementation and complexity of RBF methods in higher dimensions are essentially the same as in one dimension. Only the adaptive algorithm will need to be different.

RBF methods for time dependent PDEs enjoy large advantages in accuracy over other flexible, but low order methods, such finite differences, finite volumes, and finite elements. However, RBF methods share

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the ease of implementation and flexibility of these lower order methods. Moving grid RBF methods are easily implemented, potentially even in complex computational domains in several space dimensions. Other highly accurate spatial discretization schemes such as pseudospectral methods do not have the inherent flexibility of the RBF methods and adaptation and complex geometries are more difficult to deal with. We have applied a modification of a simple moving grid algorithm, which was developed for use with low order finite difference methods, to RBF methods for time dependent PDEs. The adaptive RBF algorithm produces excellent results.

The numerical solution of PDEs by RBF methods is based on a scattered data interpolation problem which we review in this section. Let $x_0, x_1, \ldots, x_N \in \Omega \subset \mathcal{R}^n$ be a given set of centers. A *radial basis function* is a function $\phi_i(x) = \phi(\|x - x_i\|_2)$, which depends only on the distance between $x \in \mathcal{R}^d$ and a fixed point $x_j \in \mathcal{R}^d$. Each function ϕ_j is radially symmetric about the center x_j. The radial basis function interpolation problem may be described as, given data $f_i = f(x_i),\ i = 0, 1, \ldots, N$, the interpolating RBF approximation is

$$s(x) = \sum_{i=0}^{N} \lambda_i \phi_i(x), \tag{1}$$

where the expansion coefficients, λ_i, are chosen so that $s(x_i) = f_i$. That is, they are obtained by solving the linear system

$$H\lambda = f, \tag{2}$$

where the elements of the *interpolation matrix* are $H_{i,j} = \phi(\|x_i - x_j\|_2)$, $\lambda = [\lambda_0, \ldots, \lambda_N]^{\mathrm{T}}$, and $f = [f_0, \ldots, f_N]^{\mathrm{T}}$. For the RBFs that we have considered in this work (Table 1 and Eq. (8)), the interpolation matrix can be shown to be invertible for distinct interpolation points [21,26].

A generalized interpolation problem also may be considered. The generalized interpolation problem is

$$s(x) = \sum_{i=0}^{N} \lambda_i \phi_i(x) + \sum_{k=1}^{M} b_k p_k(x), \tag{3}$$

in which a finite number of d-variate polynomials of at most order M are added to the RBF basis. The polynomials $p_k(x)$ are the polynomials spanning π_M, that is they are the polynomials of degree at most M. The extra equation(s) needed to complete the generalized interpolation problem are chosen to be

$$\sum_{j=0}^{N} \lambda_j p_k(x_j) = 0, \tag{4}$$

for $k = 1, \ldots, M$. Interpolation problem (3) must be considered when using RBFs, such as the cubics $\phi(r) = r^3$, as the basic interpolation problem (1) does not lead to a guaranteed invertible interpolation matrix [22]. Also, the generalized interpolation problem may lead to an approximation with some desirable properties that an approximation from the standard interpolation problem may lack, such as a degree of polynomial accuracy. This is the case with the multiquadric RBF [13].