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Bicritical domination

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Abstract

A graph G is domination bicritical if the removal of any pair of vertices decreases the domination number. Properties of bicritical graphs are studied. We show that a connected bicritical graph has domination number at least 3, minimum degree at least 3, and edge-connectivity at least 2. Ways of constructing a bicritical graph from smaller bicritical graphs are presented.

Keywords: Domination; Vertex critical domination; Vertex bicritical domination; Bounds; Diameter

1. Introduction

For many graph parameters, criticality is a fundamental issue. Much has been written about graphs for which a parameter (such as connectedness or chromatic number) increases or decreases whenever an edge or vertex is removed or added. For domination number, Brigham et al. [2] began the study of graphs where the domination number decreases on the removal of any vertex. Further properties of these graphs were explored in [2,3,5], but

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they have not been characterized. Other types of domination critical graphs have also been studied, for example, see [4,9–12].

In this paper, we introduce and study those graphs where the domination number decreases on the removal of any set of k vertices. Recall that for a graph $G=(V,E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v)=\{x \in V \mid vx \in E\}$. The *closed neighborhood* is $N[v]=N(v)\cup\{v\}$. A set $S \subset V$ is a *dominating set* if every vertex in V is either in S or is adjacent to a vertex in S, that is, $V=\bigcup_{s\in S} N[s]$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of minimum cardinality is called a $\gamma(G)$-set. For a set S, a vertex v is a *private neighbor of* u (with respect to S) if $N[v]\cap S=\{u\}$; and the *private neighbor set of u, with respect to S*, is the set $\mathrm{pn}[u,S]=\{v \mid N[v]\cap S=\{u\}\}$. We denote the subgraph induced by S in G by $G[S]$. We denote the distance between two vertices x and y in G by $d_G(x,y)$. For a detailed discussion of domination and for notation not defined here, see [6,7].

Note that removing a vertex can increase the domination number by more than one, but can decrease it by at most one. It is useful to write the vertex set of a graph as a disjoint union of three sets according to how their removal affects $\gamma(G)$. Let $V(G)=V^0\cup V^+\cup V^-$ where

$$V^0=\{v \in V \mid \gamma(G-v)=\gamma(G)\},$$
$$V^+=\{v \in V \mid \gamma(G-v)>\gamma(G)\},$$

and

$$V^-=\{v \in V \mid \gamma(G-v)<\gamma(G)\}.$$

It is possible for a single graph to have all of the sets V^0, V^-, and V^+ nonempty. For example, if $k\geqslant 3$ and T is the tree obtained from a star $K_{1,k}$ with center u by subdividing an edge uw of this star once, then $V^+=\{u\}$, $V^-=\{w\}$, and $V^0=V(T)-\{u,w\}$.

Brigham et al. [2] defined a vertex v to be *critical* if $v \in V^-$, and a graph G to be *domination critical* if every vertex of G is critical. A generalization of this concept was presented in [8]. Here we consider a different generalization. We define a graph G to be (γ,k)-*critical*, if $\gamma(G-S)<\gamma(G)$ for any set S of k vertices. Obviously, a (γ,k)-critical graph G has $\gamma(G)\geqslant 2$. For instance, $\overline{K}_n$ is (γ,k)-critical for all $k\leqslant n-1$. The $(\gamma,1)$-critical graphs are precisely the domination critical graphs introduced by Brigham, Chinn, and Dutton. In the special case of $k=2$, we say that G is *domination bicritical*, or just *bicritical*.

In this paper, we call a graph *critical* (respectively, *bicritical*) if it is domination critical (respectively, domination bicritical). Further, we call a graph γ-critical (respectively, γ-bicritical) if it is domination critical (respectively, γ-bicritical) with domination number γ. For example, the self-complementary Cartesian product $G=K_3\square K_3$, where $\gamma(G)=3$, is 3-critical and 3-bicritical, since removing any vertex or any pair of vertices decreases the domination number. However, critical graphs are not necessarily bicritical. For instance, the cycles C_n for $n\equiv 1 \pmod 3$ are critical, but not bicritical. On the other hand, bicritical graphs are not necessarily critical. For example, the graph H formed from the Cartesian product $K_3\square K_3$ (where the vertices of the ith copy of K_3 are labelled v_{ij} for $1\leqslant j\leqslant 3$) by adding a new vertex x adjacent to v_{11}, v_{12}, v_{23}, and v_{33} is bicritical and not critical (since $x \in V^0$).