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On the size of partial block designs with large blocks

András Sárközy[a,1], Gábor N. Sárközy[b,c,2]

[a] *Department of Algebra and Number Theory, Eötvös University, Pázmány Péter sétány 1/c, H-1117 Budapest, Hungary*
[b] *Computer Science Department, Worcester Polytechnic Institute, Worcester, MA 01609, USA*
[c] *Computer and Automation Research Institute, Hungarian Academy of Sciences, Budapest, P.O. Box 63, H-1518 Budapest, Hungary*

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Abstract

A $t-(n,k,\lambda)$ design is a k-uniform hypergraph with the property that every set of t vertices is contained in exactly λ of the edges (blocks). A partial $t-(n,k,\lambda)$ design is a k-uniform hypergraph with the property that every set of t vertices is contained in *at most* λ edges; or equivalently the intersection of every set of $\lambda+1$ blocks contains fewer than t elements. Let us denote by $f_\lambda(n,k,t)$ the maximum size of a partial $t-(n,k,\lambda)$ design. We determine $f_\lambda(n,k,t)$ as a fundamental problem in design theory and in coding theory. In this paper we provide some new bounds for $f_\lambda(n,k,t)$.

Keywords: Partial block designs

1. Introduction

1.1. Notation and definitions

A *hypergraph* H is a set $V(H)$, whose elements are called vertices, and a set $E(H)$ of subsets of $V(H)$, whose elements are called edges. A hypergraph is *k-uniform* if each of its edges contains exactly k vertices.

E-mail addresses: sarkozy@cs.elte.hu (A. Sárközy), gsarkozy@cs.wpi.edu (G.N. Sárközy).

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A $t-(n,k,\lambda)$ *design* is a k-uniform hypergraph on n vertices with the property that every set of t vertices is contained in exactly λ of the edges (blocks). A *partial* $t-(n,k,\lambda)$ *design* is a k-uniform hypergraph on n vertices with the property that every set of t vertices is contained in *at most* λ edges; or equivalently the intersection of every set of $\lambda+1$ blocks contains fewer than t elements. (Partial) $t-(n,k,1)$ designs are often called (partial) *Steiner systems*. Let us denote by $f_\lambda(n,k,t)$ the maximum size of a partial $t-(n,k,\lambda)$ design.

A *binary code* C consists of bit vectors (codewords) of length m, where the *weight* $w(\mathbf{e})$ of a vector $\mathbf{e}$ is equal to the number of ones in $\mathbf{e}$. If $\mathbf{x}$ and $\mathbf{y}$ are codewords, then their *distance* $d(\mathbf{x},\mathbf{y})$ is the number of places where they differ, i.e. $d(\mathbf{x},\mathbf{y})=w(\mathbf{x}-\mathbf{y})$. Then the *minimum distance* of code C is the minimal value of $d(\mathbf{x},\mathbf{y})$ for all pairs of distinct codewords.

In this paper $\log n$ denotes the natural logarithm. We will also use the standard notation $\mathrm{e}(\alpha)=\mathrm{e}^{2\pi\mathrm{i}\alpha}$.

1.2. On the size of block designs

Determine $f_\lambda(n,k,t)$ as a fundamental problem; it is studied in various forms. It is sometimes also referred to as the packing problem for hypergraphs. It is also related to the Turán problem for hypergraphs, the lottery problem, the football pool problem, etc. It is also a fundamental problem in coding theory. Determine $f_1(n,k,t)$ as equivalent to finding the maximum size $A(n,d,k)$ of a binary code with word length n, constant weight k and minimum distance $d \geqslant 2(k-t+1)$.

Since this is such a fundamental question, there is a vast literature on results concerning $f_\lambda(n,k,t)$ (see e.g. [2,7] or [9]). The majority of the results concentrate on the case when k and t are "small". For example, simple counting shows that for the number of edges in any partial Steiner system we have

$$f_1(n,k,t) \leqslant \frac{\binom{n}{k}}{\binom{k}{t}}.$$

In a breakthrough paper [10], Rödl developed the nibble technique in order to prove the conjecture of Erdős and Hanani [3], which asserts that for every fixed $k \geqslant t > 0$ and $n \to \infty$ we have

$$f_1(n,k,t) = (1-\mathrm{o}(1))\frac{\binom{n}{k}}{\binom{k}{t}}.$$

In this paper we are interested in the other extreme, namely when k is large: $\varepsilon n < k \leqslant (1-\varepsilon)n$ for some $\varepsilon > 0$. We will define c by $k=cn$ so that $0<c<1$. In particular, we are interested in what happens when t is around the "expected value" of the intersection of $\lambda+1$ blocks. Assuming that we selected the $\lambda+1$ blocks randomly, this expected value is $c^{\lambda+1}n$.

Fundamental results in coding theory (see [6]) imply that for $k=cn$, the function $f_1(n,k,t)$ is polynomial in n for $t \leqslant c^2 n + 1$, and exponential for $t \geqslant (c^2+\varepsilon)n$ for any $\varepsilon > 0$. In addition, the well-known Johnson bound (see [5] or [6]) gives the following upper