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Diagonal flips in Hamiltonian triangulations on the projective plane

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Abstract

In this paper, we shall prove that any two triangulations on the projective plane with n vertices can be transformed into each other by at most $8n - 26$ diagonal flips, up to isotopy. To prove it, we focus on triangulations on the projective plane with contractible Hamilton cycles.

Keywords: Diagonal flip; Triangulation; Projective plane; Hamiltonian cycle

1. Introduction

For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A *k-cycle* means a cycle of length k. For two graphs H and K, let $H + K$ denote the graph obtained from H and K by joining each vertex of H to all vertices of K.

A *triangulation* on a closed surface F^2 is a simple graph on F^2 such that each face is bounded by a 3-cycle. A *diagonal flip* is an operation which replaces an edge e in the quadrilateral D formed by two faces sharing e with another diagonal of D (see Fig. 1). If the resulting graph is not simple, then we do not apply it.

Wagner proved that any two triangulations on the plane with the same number of vertices can be transformed into each other by a sequence of diagonal flips, up to isotopy [9]. This result has been extended to the torus [1], the projective plane and the Klein bottle [7]. Moreover, Negami has proved that for any closed surface F^2, there exists a positive integer

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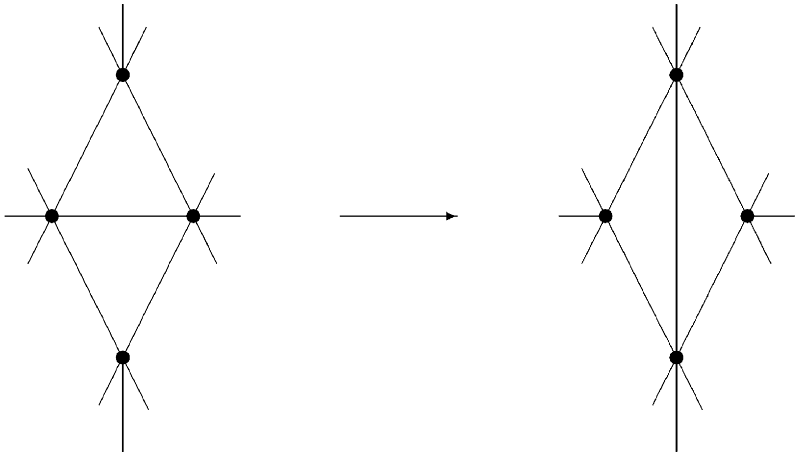

Fig. 1. Diagonal flip.

$N(F^2)$ such that any two triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geqslant N(F^2)$ can be transformed into each other by a sequence of diagonal flips, up to homeomorphism [4]. There are many papers concerning with diagonal flips in triangulations and they are described in [6] for details.

In this paper, we focus on the minimum number of diagonal flips needed to transform two given triangulations on a closed surface F^2. Negami's argument in [5] shows that for the minimum number of diagonal flips needed to transform two triangulations with n vertices on a closed surface F^2, there is a quadratic bound with respect to n. However, if we restrict F^2 to the sphere, then there is a linear bound $6n - 30$ for it, as shown in [3].

In this paper, we shall prove the following theorem:

Theorem 1. *Any two triangulations on the projective plane with n vertices can be transformed into each other by at most* $8n - 26$ *diagonal flips*, *up to isotopy*.

This is the first result giving a linear bound for the minimum number of diagonal flips in triangulations on a closed surface other than the sphere.

For a graph G, a *Hamilton cycle* of G is a cycle passing through each vertex of G exactly once. A cycle C of G embedded in a closed surface F^2 is said to be *contractible* if C bounds a 2-cell on F^2. In order to prove Theorem 1, we show the following theorem for triangulations on the projective plane with a contractible Hamilton cycle, as in the spherical case in [3].

Theorem 2. *Let G and G′ be two triangulations on the projective plane with n vertices*, *each of which has a contractible Hamilton cycle. Then G and G′ can be transformed into each other by at most* $6n - 12$ *diagonal flips*, *preserving their Hamilton cycles*.

2. Triangulations with contractible Hamilton cycles

In this section, we deal only with triangulations which have contractible Hamilton cycles. Clearly, a contractible Hamilton cycle in a triangulation G on the projective plane separates