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Note

A bound on the total size of a cut cover

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Abstract

A *cycle cover* (*cut cover*) of a graph G is a collection of cycles (cuts) of G that covers every edge of G at least once. The *total size* of a cycle cover (cut cover) is the sum of the number of edges of the cycles (cuts) in the cover.

We discuss several results for cycle covers and the corresponding results for cut covers. Our main result is that every connected graph on n vertices and e edges has a cut cover of total size at most $2e - n + 1$ with equality precisely when every block of the graph is an odd cycle or a complete graph (other than K_4 or K_8). This corresponds to the result of Fan [J. Combin. Theory Ser. B 74 (1998) 353–367] that every graph without cut-edges has a cycle cover of total size at most $e + n - 1$.

Keywords: Cut; Cycle; Edge cover

1. Cycle covers and cut covers

Covering the edges of a graph by subgraphs from a given family of graphs, like cliques, matchings, trees, cycles, or cuts is one of the basic themes in graph theory (see Pyber [21] for a survey of results). Erdős et al. [5] showed that the edges of every graph on n vertices can be covered by $\lfloor n^2/4 \rfloor$ cliques, and the balanced complete bipartite graph shows that this is best possible. It can also be desirable to minimize parameters other than the number of subgraphs used in the cover. Győri and Kostochka [11], Chung [4] and Kahn [17] independently proved the stronger result that every graph has a decomposition into

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cliques whose *order-sum* (sum of the number of vertices of the cliques in the cover) is at most $\lfloor n^2/2 \rfloor$.

A heavily studied edge covering concept is that of a cycle cover: A *cycle cover* of a graph G is a collection of cycles of G such that every edge of G is in at least 1 cycle. Since an edge of a graph is in a cycle precisely when it is not a cut-edge, cycle covers exist only for graphs without cut-edges (also called *bridgeless* graphs). One of the outstanding questions on cycle covers is the Cycle Double Cover Conjecture (CDCC) of Seymour [22] and Szekeres [23]: Every bridgeless graph has a cycle cover such that every edge is in exactly 2 cycles. This question is connected to several topological questions. The interested reader should consult the book of Zhang [26] for more information on the CDCC and other cycle cover problems.

The *total size* of a cycle cover is the sum of the number of edges over all cycles in the cover. Thus, a cycle double cover of a graph on e edges has total size $2e$. The minimum total size of a cycle cover of a bridgeless graph G is denoted by $\mathrm{scc}(G)$ (for shortest cycle cover). There are a number of interesting questions on shortest cycle covers of graphs. In 1995, Thomassen [24] settled a conjecture of Itai et al. [12] by proving that the problem of determining $\mathrm{scc}(G)$ is NP-complete. A conjecture of Alon and Tarsi [1] claims that every bridgeless graph on e edges satisfies $\mathrm{scc}(G) \leqslant \frac{7}{5}e$ with equality for the Petersen graph and various graphs derived from it. Interestingly, Jamshy and Tarsi [16] proved that this conjecture implies the CDCC.

A *cut* induced by a set of vertices S consists of all edges with exactly one endpoint in S. The notions of a *cut cover* of a graph G, and the minimum total size of such a cut cover, denoted by $\mathrm{ccs}(G)$ (for cut cover size) can now be defined similarly. For a bridgeless graph G embedded in the plane, the dual of a cycle forms a cut in the dual graph G^*, so that it is easy to see that $\mathrm{scc}(G) = \mathrm{ccs}(G^*)$. Considering this duality it seems reasonable to expect that similarly intriguing questions arise when studying cut covers. It turns out that in several ways cuts behave nicer than cycles. For one, every graph *has* a cut cover.

A *star cut* is a cut in which the set S has size 1. This notion leads immediately to a "Cut Double Cover Theorem": if a cut cover consists of all the star cuts of a graph, then every edge is covered exactly twice. Hence $\mathrm{ccs}(G) \leqslant 2e(G)$. Unfortunately, however, determining $\mathrm{ccs}(G)$ is still an NP-complete problem, even when restricted to graphs with maximum degree 3 (see [10]).

The focus of this paper is the dual question to the following cycle cover question of Itai and Rodeh [13]: Does every bridgeless graph on n vertices and e edges have a cycle cover of total size at most $e + n - 1$, i.e. is $\mathrm{scc}(G) \leqslant e + n - 1$? After a flurry of papers [13,12,1,3,9,6,7,2] this question was finally settled in the affirmative by Fan [8]. By Euler's formula we thus obtain that for a (loopless) connected plane graph with n vertices, e edges and f faces

$$\mathrm{ccs}(G) = \mathrm{scc}(G^*) \leqslant e + f - 1 = e + (2 + e - n) - 1 = 2e - n + 1.$$

Our main result is that this bound holds for non-planar graphs as well and we characterize the cases of equality: