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Note

A note on internally disjoint alternating paths in bipartite graphs ☆

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Abstract

Let G be a balanced bipartite graph with partite sets X and Y, which has a perfect matching, and let $x \in X$ and $y \in Y$. Let k be a positive integer. Then we prove that if G has k internally disjoint alternating paths between x and y with respect to some perfect matching, then G has k internally disjoint alternating paths between x and y with respect to every perfect matching.

Keywords: Matching; Alternating path

For graph-theoretic terminology not defined in this note, we refer the reader to [2]. In this note, a path which starts from a vertex x and ends at a vertex y is called an xy-path.

For a matching M of a graph G, a trail $T = a_0 a_1 \dots a_l$ is said to be an alternating trail with respect to M if $a_{2i-1}a_{2i} \in M$ for each i with $1 \leqslant i \leqslant \lfloor \frac{1}{2} l \rfloor$. Note that this implies $a_{2j}a_{2j+1} \in E(G) - M$ $(0 \leqslant j \leqslant \lfloor \frac{l-1}{2} \rfloor)$. If T is a path (resp. cycle), we call T an alternating path (resp. alternating cycle). Note that by the definition, if P is an alternating xy-path of even length, the same path traversed in the opposite direction is not an alternating yx-path since the first edge belongs to the perfect matching.

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For a nonnegative integer n and a graph G of order at least $2n+2$, G is said to be n-extendable if G has a perfect matching and every set of n independent edges extends to a perfect matching. The reader can be referred to [3]. In [1], Aldred et al. have found the following Menger-type relationship between the extendability and the number of internally disjoint alternating paths between a pair of vertices in a bipartite graph.

Theorem A (*Aldred et al. [1]*). *Let G be a balanced bipartite graph with partite sets X and Y which has a perfect matching, and let k be a positive integer. Then G is k-extendable if and only if for every perfect matching M and every pair of vertices $x \in X$ and $y \in Y$, there exist k internally disjoint xy-paths with respect to M in G.*

The sufficiency of Theorem A requires us to check every perfect matching. However, the following theorem claims that we have only to check just one perfect matching which is arbitrarily chosen, since it forces the existence of k internally disjoint alternating paths for all the other perfect matchings.

Theorem 1. *Let G be a balanced bipartite graph with partite sets X and Y, which has a perfect matching, and let $x \in X$ and $y \in Y$. Let M_0 and M be perfect matchings of G. If G has k internally disjoint alternating xy-paths with respect to M_0, then G has k internally disjoint alternating xy-paths with respect to M.*

For a vertex x in a graph G, we denote by $N_G(x)$ the neighborhood of x in G. In this note, we deal with a matching as a set of edges. For example, if an edge e belongs to a matching M, we write $e \in M$. Let $T = v_0v_1 \ldots v_l$ be a trail. For indices i and j with $0 \leqslant i < j \leqslant l$, the subtrail $v_iv_{i+1} \ldots v_j$ is denoted by $v_i \overrightarrow{T} v_j$. The same subtrail, traversed in the opposite direction, is denoted by $v_j \overleftarrow{T} v_i$. For two sets A and B, $A \triangle B$ denotes the symmetric difference of A and B.

Proof of Theorem 1. Let $P_1, \ldots, P_k$ be internally disjoint alternating xy-paths with respect to M_0. Let $H = (V(G), \bigcup_{i=1}^{k} E(P_i))$. Then for $v \in V(H) = V(G)$, we have

$$\deg_H v = \begin{cases} 0 & \text{if } v \notin \bigcup_{i=1}^{k} V(P_i) \\ 2 & \text{if } v \in \bigcup_{i=1}^{k} V(P_i) - \{x, y\} \\ k & \text{if } v = x \text{ or } v = y. \end{cases}$$

Let $K = (V(G), E(H) \triangle M_0 \triangle M)$. First, we investigate the degree of the vertices in K. $\square$

Claim 1. *For each $v \in V(G) - \bigcup_{i=1}^{k} V(P_i)$, $\deg_K v = 0$ or $\deg_K v = 2$. Furthermore, if $\deg_K v = 2$, then exactly one of the two edges of K incident with v belongs to M.*

Proof. Since both M_0 and M are perfect matchings, $vv_0 \in M_0$ and $vv' \in M$ for some $v_0, v' \in N_G(v)$. Furthermore, since $v \notin \bigcup_{i=1}^{k} V(P_i)$, no edge of H is incident with v. If $v' = v_0$, then $vv' = vv_0 \in (M_0 \cap M) - E(H)$, which implies $vv' \notin E(K)$ and $\deg_K v = 0$. If $v' \neq v_0$, then $vv_0 \in M_0 - (M \cup E(H))$ and $vv' \in M - (M_0 \cup E(H))$. Hence we have $N_K(v) = \{v_0, v'\}$ and $\{vv_0, vv'\} \cap M = \{vv'\}$. $\square$