Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Mathematics 301 (2005) 34–45

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Partially balanced incomplete block designs from weakly divisible nearrings☆

A. Benini*, F. Morini

Dipartimento di Matematica, Facoltà di Ingegneria, Università degli Studi di Brescia, Via Valotti 9, I-25133 Brescia, Italy

Received 31 July 2002; received in revised form 21 September 2004; accepted 7 March 2005
Available online 18 August 2005

Abstract

In [[6] Riv. Mat. Univ. Parma 11 (2) (1970) 79–96] Ferrero demonstrates a connection between a restricted class of planar nearrings and balanced incomplete block designs. In this paper, bearing in mind the links between planar nearrings and weakly divisible nearrings (wd-nearrings), first we show the construction of a family of partially balanced incomplete block designs from a special class of wd-nearrings; consequently, we are able to give some formulas for calculating the design parameters.

Keywords: Block design; Association scheme; Nearring

1. Introduction

A nearring N is called a *weakly divisible nearring* (wd-nearring) if the following condition is satisfied: $\forall a, b \in N \ \exists x \in N | ax = b$ or $bx = a$. This algebraic structure was first defined and studied in [4] and a method to construct a special class of wd-nearrings was found in [2,3]. This method has been generalized and implemented in " SONATA", a package of GAP [1].

The structure of a finite wd-nearring is quite similar to that of a better known planar nearring. Since planar nearrings have been a powerful tool in the construction of balanced

☆ Work carried out on behalf of Italian M.I.U.R.

* Corresponding author.
E-mail address: anna.benini@ing.unibs.it (A. Benini).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.03.026

incomplete block designs (BIB-designs), in this paper it is shown that partially balanced incomplete block designs (PBIB-designs) can be constructed from a class of wd-nearrings. More precisely, the paper is organized as follows:

In Section 2 we gather some results on wd-nearrings which we will use throughout the paper.

In Section 3, using the structure and the properties of a suitable class of wd-nearrings, we show how it is possible to construct block designs and to compute their parameters.

In Section 4 we firstly recall that, starting from an orbital design and using a general construction of Hall, it is possible to define an association scheme making the design a PBIB-design. Then, we prove that the previously constructed designs are orbital designs or a disjoint union of them. Thus, such designs become partially balanced and several formulas to compute their parameters are proved.

In Section 5, to facilitate the application of the many steps of the whole construction, we will conclude showing an example.

2. Weakly divisible nearrings

A *left nearring* is an algebraic structure $N = (S, +, *)$ such that $(S, +)$ is an additive group, $(S, *)$ is a multiplicative semigroup, and the left distributive low holds (see [5,9]). In the sequel we always consider left *zerosymmetric* nearrings, that is, $0 * x = 0,\ \forall x \in N$.

In this section, we shall summarize the results, terminology and notation from [4,2,3] that will be used in the sequel.

Definition 2.1. A nearring *N* is called weakly divisible (wd-nearring) if the following condition is satisfied:

$$\forall a, b \in N\ \exists x \in N \mid ax = b \text{ or } bx = a.$$

In [4] it is proved that a finite wd-nearring *N* is the disjoint union of the nil radical *Q* (the set of the nilpotent elements of *N*) and the multiplicative semigroup *C* of the left cancellable elements. Moreover, by Theorem 8 of [4], the set *C* is the disjoint union of its maximal multiplicative subgroups, isomorphic to each other. As in [2], γ_a denotes the left translation defined by *a*, for $a \in N$, that is, $\gamma_a(x) = ax$, for every $x \in N$. We know that γ_a is an endomorphism of N^+ which turns out to be an automorphism if, and only if, *a* is a left cancellable element of *N*. Furthermore, by Proposition 2 of [2] we note that $\Gamma(C)$, the set of the left translations defined by the elements of *C*, is an automorphism group of N^+ with respect to composition, and the fixed points of $\gamma_c \neq id_N$, $c \in C$, are nilpotent elements of *N*.

Definition 2.2. Let *p* be a prime number and consider the residue class group (*modulo* p^n) $(\mathbb{Z}_{p^n}, +)$. A p^n-maximal wd-nearring *N* is a finite wd-nearring on $(\mathbb{Z}_{p^n}, +)$, in which the set *Q* of the nilpotent elements of *N* coincides with $p\mathbb{Z}_{p^n}$.

Obviously, the ring $\mathbb{Z}_{p^n}$ is, in particular, a p^n-maximal wd-nearring but this trivial case will be excluded in the following.