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Distance-j ovoids and related structures in generalized polygons

Alan Offer, Hendrik Van Maldeghem

Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B 9000 Ghent, Belgium

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Abstract

Given a finite weak generalized polygon Γ with an order (s, t), we provide necessary conditions on the order for Γ to admit a distance-j ovoid with odd j. This leads to the introduction and study of similar structures involving flags, which we name *floveads*.

MSC: 51E12; 51E20

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1. Introduction

A *weak generalized n-gon* is a geometry $\Gamma = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ of points and lines whose incidence graph has diameter n and girth $2n$. If each line of Γ is incident with exactly $s+1$ points and each point is incident with exactly $t+1$ lines, then Γ has order (s, t), and if $s = t$ then we may also say that Γ has order s. If both $s, t \geqslant 2$ then Γ is a *generalized n-gon*. By Feit and Higman [4], apart from ordinary n-gons, finite weak generalized n-gons with $n > 2$ and having an order (s, t) can exist only for $n \in \{3, 4, 6, 8, 12\}$, and if $n = 12$ then either $s = 1$ or $t = 1$.

E-mail addresses: aoffer@cage.UGent.be (A. Offer), hvm@cage.UGent.be (H. Van Maldeghem).

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The distance $\delta(u, v)$ between two elements u and v of Γ is the distance between them in the incidence graph. In particular, the value of $\delta(u, v)$ is at most n, the diameter of the incidence graph, and when $\delta(u, v) = n$ we say the elements u and v are *opposite*. For an element u of Γ, the set of elements at distance d from u is denoted $\Gamma_d(u)$. The sizes of the sets $\Gamma_d(u)$, $\mathscr{P}$ and $\mathscr{L}$ are given in [13, Lemma 1.5.4]. In particular, we will use $|\Gamma_{2i}(p)| = s^i t^{i-1}(t+1)$, where p is a point and $0 < 2i < n$, and $|\mathscr{P}| = (1+s)(1+st+s^2t^2+\cdots+s^{m-1}t^{m-1})$, when $n = 2m$ is even.

The dual Γ^D of a weak generalized n-gon $\Gamma = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ is the incidence structure $\Gamma^D = (\mathscr{L}, \mathscr{P}, \mathrm{I})$ obtained by interchanging the roles of points and lines. The dual Γ^D is then also a weak generalized n-gon, and if Γ has order (s, t) then Γ^D has order (t, s).

The *double* 2Γ of a weak generalized n-gon $\Gamma = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ is the incidence structure obtained by taking as points the points and lines of Γ, and as lines the flags $\{p, L\}$, $p \in \mathscr{P}$, $L \in \mathscr{L}$, of Γ, with incidence being symmetrized inclusion. This is really just the incidence graph of Γ with vertices and edges considered as points and lines. The double 2Γ is a weak generalized $2n$-gon, and if Γ has order s then 2Γ has order $(1, s)$. In fact, every finite weak generalized $2n$-gon with order $(1, s)$ arises as the double of a weak generalized n-gon of order s ([12], see also [13, 1.6.2]).

Let Γ be a weak generalized n-gon. For $1 \leqslant j \leqslant n/2$, a *distance-j ovoid* is a set $\mathscr{O}$ of points such that any two points of $\mathscr{O}$ are at least distance $2j$ apart and such that for every element p of Γ there is some element $q \in \mathscr{O}$ with $\delta(p, q) \leqslant j$. The dual notion is that of a *distance-j spread*. When $j = n/2$, we speak simply of *ovoids* and *spreads*.

There are several factors that motivate the study of distance-j ovoids in finite weak generalized n-gons. For instance, they give rise to perfect codes when j is odd (see [2]), and they are related to epimorphisms from n-gons to m-gons with $n \neq m$ (see [5,6]). They also have relationships with such objects as 1-systems, semipartial geometries and strongly regular graphs (see [9]).

In Section 2, we give necessary conditions for the existence of certain distance-j ovoids.

2. Distance-j ovoids

In [10], it is shown that a finite weak generalized hexagon Γ of order (s, t) can have an ovoid $\mathscr{O}$ only if $s = t$. This is done there by a double counting argument, first by counting the points that lie in 'neighbourhoods' of the points of $\mathscr{O}$ and then by fixing a point p of $\mathscr{O}$ and counting the other points of $\mathscr{O}$ according to their positions relative to p. Applying this same idea to distance-j ovoids in other finite weak generalized $2m$-gons with an order (s, t) tells us nothing new when j is even as the two counts result in the same expression, but we do get restrictions on the order when j is odd. That is to say, in light of Feit and Higman [4], that this approach yields results when (j, m) is (3, 3), (3, 4), (3, 6) or (5, 6). As the first of these cases is treated by [10], here we treat the remaining ones and so prove the following theorem.

Theorem 1. *If a finite weak generalized octagon of order (s, t) admits a distance-3 ovoid then $s = 2t$. If a finite weak generalized dodecagon of order (s, t) admits a distance-3 ovoid then (s, t) is either $(1, 1)$ or $(3, 1)$. No finite weak generalized dodecagon with an order (s, t) has a distance-5 ovoid.*