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Note

The crossing number of the circular graph $C(2m+2, m)$ ☆

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Abstract

The circular graph $C(n, m)$ is such a graph that whose vertex set is $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and edge set is $\{v_i v_{i+1}, v_i v_{i+m} \mid i = 0, 1, \ldots, n-1\}$, where m, n are natural numbers, addition is modulo n, and $2 \leqslant m \leqslant \lfloor n/2 \rfloor$. This paper shows the crossing number of the circular graph $C(2m+2, m)(m \geqslant 3)$ is $m+1$.

Keywords: Crossing number; Circular graph; Automorphism

1. Introduction

Let G be a simple graph with the vertex set V and the edge set E. A *drawing* of G in the plane $\mathscr{R}^2$ is an immersion $\phi : G \to \mathscr{R}^2$ such that
(1) $\phi(v) \cap \phi(x) = \emptyset$ for each $v \in V(G)$ and $x \in (V(G) \cup E(G)) - \{v\}$, and
(2) $\phi(e) \cap \phi(f)$ is finite for each pair $\{e, f\}$ of edges of G.

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The drawing is called *good*, if for all $\phi(E)$, no one crosses itself, no two cross more than once, and no more than two cross at a point in the plane. A *crossing* in a good drawing is a point of intersection of two elements in $\phi(E)$. A good drawing is said to be *optimal* if it minimizes the number of crossings. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the number of crossings in any optimal drawing of G in the plane.

The circular graph $C(n, m)$ is such a graph that whose vertex set is $\{v_0, v_1, v_2, \dots, v_{n-1}\}$ and edge set is $\{v_i v_{i+1}, v_i v_{i+m} \mid i = 0, 1, \dots, n-1\}$, where m, n are natural numbers, addition is modulo n, and $2 \leqslant m \leqslant \lfloor n/2 \rfloor$. It can be seen that $C(3, 2) = K_3$, $C(4, 2) = K_4$, and $C(5, 2) = K_5$. When $2 \leqslant m < n/2$, the circular graph $C(n, m)$ is a minor of the generalized Petersen graph $G(n, m)$. The generalized Petersen graph $G(n, m)$ is the graph which vertex set is $\{u_i, v_i \mid i = 0, 1, \dots, n-1\}$ and edge set is $\{u_i u_{i+1}, v_i v_{i+m}, u_i v_i \mid i = 0, 1, \dots, n-1\}$, where m, n are natural numbers, addition is modulo n, and $m < n/2$. $G(n, m)$ has been studied in many contexts. Its crossing number is an interesting object [1,4,6]. The crossing number of $C(n, m)$ was initially investigated in papers [3] and [5], which gave an upper bound on the crossing number of $C(n, m)$. Since $C(2m + 2, m)$ is planar for $m = 2$, we always assume $m \geqslant 3$ in this paper, and will prove $\mathrm{cr}(C(2m + 2, m)) = m + 1$.

For graph theory terminology we refer to [2].

2. The main result

Let $\mathscr{C}_{2m+2} = (v_0 v_{2m+1} v_{2m}, \dots, v_{m+3} v_1 v_2 v_3, \dots, v_{m+1} v_{m+2})$, which is Hamiltonian cycle of $C(2m + 2, m)$. Draw it in the plane with $m + 1$ vertices $v_0, v_{2m+1}, v_{2m}, \dots, v_{m+4}, v_{m+3}, v_1$ in a column, and $m + 1$ vertices $v_{m+2}, v_{m+1}, v_m, \dots, v_4, v_3, v_2$ in another column. Add the other edges according to the definition of $C(2m+2, m)$. A drawing of $C(2m+2, m)$ in the plane is shown in Fig. 1 . It can be seen from Fig. 1 that there are m edges parallel with the edge $v_1 v_2$, and each but $v_0 v_{m+2}$ has only one crossing. Combining with the other two crossings, the following result is obvious.

Lemma 2.1. $\mathrm{cr}(C(2m + 2, m)) \leqslant m + 1$.

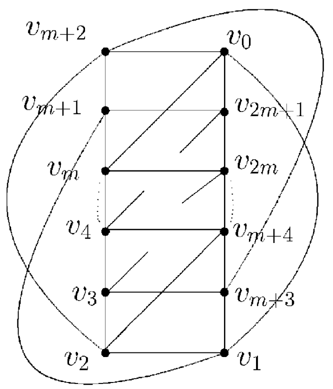

Fig. 1.