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On matroids determined by their Tutte polynomials

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Abstract

A matroid is T-unique if it is determined up to isomorphism by its Tutte polynomial. Known T-unique matroids include projective and affine geometries of rank at least four, wheels, whirls, free and binary spikes, and certain generalizations of these matroids. In this paper we survey this work and give three new results. Namely, we prove the T-uniqueness of $M(K_{m,n})$ and of the truncations of $M(K_n)$, and we show the existence of exponentially large families of T-unique matroids.

1. Introduction

The Tutte polynomial $t(M; x, y)$ of a matroid M is a powerful invariant that encodes a considerable amount of information about M. From $t(M; x, y)$ one can obtain many important invariants of the matroid, including the characteristic polynomial, the rank, the number of bases, and the number of connected components. For a graphic matroid, $M = M(G)$, the Tutte polynomial contains as specializations the chromatic and the flow polynomials of G. In addition to other applications to graph theory, the Tutte polynomial also has connections with knot theory, coding theory, and statistical mechanics (see [10,25] for useful surveys).

A question that arises naturally is that of whether a matroid is determined up to isomorphism by the information contained in its Tutte polynomial. This motivates the following definition.

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Definition 1.1. Two matroids M and N are T-equivalent if $t(M; x, y) = t(N; x, y)$. A matroid M is *T-unique* if every matroid N that is T-equivalent to M is isomorphic to M.

Using the bounds that appear in [10, Exercise 6.9], it is easy to show that almost all matroids are not T-unique. In this paper, we shall focus precisely on matroids that have been proved to be T-unique. Our aim is both to survey previous results and techniques, and to present new families of T-unique matroids.

The related problem of finding large families of T-equivalent nonisomorphic matroids has also received much attention. The first examples known were pairs of T-equivalent graphic matroids given by Tutte [24], and later by Brylawski [8]. In [2] Bollobás, Pebody, and Riordan give a method for constructing exponentially large families of T-equivalent highly connected graphic matroids, where exponentially large means that the size of the family grows exponentially as a function of the rank. In [3], Bonin uses inequivalent representations of matroids over finite fields to produce many sequences of exponentially large families of 3-connected T-equivalent representable matroids.

The problem of T-uniqueness can also be posed within the class of 2-connected graphs. In [16,19] certain graphs are shown to be determined among all graphs by their Tutte polynomials. This work can be viewed as an extension of the search for graphs determined by their chromatic polynomials (see [13,14] for a survey on this problem). However, knowing that a graph is determined by its Tutte polynomial in this sense does not imply the T-uniqueness of the graphic matroid as defined above (see [10, Section 2] for a graphic and a nongraphic matroid that have the same Tutte polynomial).

The outline of this paper is as follows. In Section 2 we define the Tutte polynomial and give a list of some of the invariants of a matroid M that can be deduced from $t(M; x, y)$. The next two sections are devoted to known families of T-unique matroids. In Section 3 we survey the results on projective and affine geometries, Dowling geometries and related matroids. In Section 4 we introduce k-chordal matroids and techniques for proving their T-uniqueness; these techniques apply to wheels, whirls, spikes, and generalizations of these matroids. In Section 5, we give new applications of the techniques developed in Section 4; specifically, we prove the T-uniqueness of the cycle matroids of complete bipartite graphs and the truncations of the cycle matroids of complete graphs. In Section 6, we give an exponentially large family of T-unique graphic matroids that arise from the cycle matroids of a family of outerplanar graphs. This family stands in contrast to the examples mentioned above of exponentially large families of T-equivalent matroids, and also to the fact that so far only few examples were known of T-unique matroids of the same rank and size.

Proving that a matroid is T-unique typically provides a characterization of this matroid in terms of some of the invariants listed in the next section. Whenever it is feasible, we give this characterization rather than only stating the T-uniqueness.

Our notation follows [20]. Recall that the *girth* $g(M)$ of a matroid M that is not free is the size of the smallest circuit of M. A matroid is called a *geometry* (or combinatorial geometry) if it is a simple matroid. A *line* is a flat of rank two, and a *plane* is a flat of rank three. For brevity, we abuse notation by saying that a flat F of M is isomorphic to N if $M|F \cong N$.