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The chip-firing game

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Abstract

The process called the chip-firing game has been around for no more than 20 years, but it has rapidly become an important and interesting object of study in structural combinatorics. The reason for this is partly due to its relation with the Tutte polynomial and group theory, but also because of the contribution of people in theoretical physics who know it as the (Abelian) sandpile model.

Here, we survey some of the numerous connections that the chip-firing game has with some other parts of combinatorics and with theoretical physics. Among these we present its relation with the Tutte polynomial, group theory, greedoids with repetition and matroids. We also reintroduce it as the Abelian sandpile model of statistical mechanics and give a relation with the Potts model.

Keywords: Chip-firing game; Tutte polynomial; Matroid

1. Introduction

In 1986, Spencer [40] was studying the following problem: given $k \in \mathbb{N}$ and n vectors $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^m$, $\|v_i\|_\infty \leqslant 1$, $0 \leqslant i \leqslant n$, do there exist $e_1, \ldots, e_n$, $e_i \in \{-1, 1\}$ for $1 \leqslant i \leqslant n$, such that $\| \sum_{i=1}^{n} e_i v_i \|_\infty \leqslant kn^{1/2}$?

His answer uses a “balancing game”. We have a pile of N chips in the center of a long path, at each move we take $\lceil N/2 \rceil$ chips to the right and $\lfloor N/2 \rfloor$ to the left. Now, the game continues with these two new piles and so on. Later, Anderson et al. in [1]

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extended the game by allowing the player to move one chip at a time either to the right or to the left, and starting with an arbitrary distribution of chips.

In 1991, Björner et al. [12] studied the natural generalization of this game to simple graphs. We put some chips on each vertex of G; we say that a vertex is *ready* if it has at least as many chips as its degree, in which case we can fire it and the result is that it distributes one chip to each of its neighbours, this can cause another vertex to be ready, and so on. This game was called the *chip-firing game*. Here, they were mainly interested in the final distribution of chips after a sequence of moves that follow the rules and the duration of the process. A game following the rules is called a legal game. Even though they were working in the context of simple graphs, exactly the same analysis can be done for graphs with multiple edges, and we state their main results in this more general context.

Theorem 1 (*Björner et al. [12]*). *Given a connected multi-graph and an initial distribution of chips, either every legal game can be continued indefinitely, or every legal game terminates after the same number of moves with the same final position. The number of times a given vertex is fired is the same in every legal game.*

Theorem 2 (*Björner et al. [12]*). *Given a connected multi-graph G and an initial distribution of N chips*:

(1) *If $N > 2|E(G)| - |V(G)|$, then the game is infinite.*
(2) *If $|E(G)| \leqslant N \leqslant 2|E(G)| - |V(G)|$, then there exists an initial configuration which terminates after a finite number of firings and also one which continues indefinitely.*
(3) *If $N < |E(G)|$, the game is finite.*

From about the same time as Spencer's studies we have the work of Bak et al. and Dhar [3,23]. In [3] they introduce the notion of *self-organized critical phenomena*. This notion has as a model *Abelian sandpiles* that were introduced by Dhar [23]. Later we will see that the chip-firing game and the Abelian sandpile model are, under certain restrictions, the same process, which is interesting as they were proposed independently and for totally different reasons.

It was Biggs [6] who came up with a process that was related to the chip-firing game and to Abelian sandpiles. In this game we also have a graph G, but this time we are given a special vertex q. The rules of this new game are as above for every vertex except for q, but q has a debit of chips equal to the number of chips on the graph and is ready only when every other vertex is not, then q is fired until some vertex is ready. The last rule ensures an infinite game. In [6], the game is called the *dollar game*, and dollars are used instead of chips. The vertex q plays the role of the government and the whole game is a simulation of the economy. Here, however, we stick to the term chip-firing game for this new game.

In Section 2 we introduce matroids and also the Tutte polynomial, these notions will be required later. Section 3 is devoted to the chip-firing game and some of its many properties, so it is a long section. We start with the definition of the chip firing and critical configuration to move later to some greedoid theory in Section 3.1, this tool is used to analyse critical configurations in Section 3.2. In Section 3.3 we include the definition of the *critical group* of a graph, and in Sections 3.4 and 3.5 we describe the relation of the chip firing and