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Knots and links in spatial graphs: A survey

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Abstract

A spatial representation, $\mathscr{R}(G)$, of a graph G, is an embedded image of G in $\mathbb{R}^3$. A set of cycles in $\mathscr{R}(G)$ can be thought of as a set of simple closed curves in $\mathbb{R}^3$ and thus they may be regarded as a *link* in $\mathbb{R}^3$. A recent area of research investigates the dependence (or independence) of the link types on the structure of the abstract graph G itself rather than on specific spatial representations of G. In this article, we survey what is known today.

Keywords: Knot; Link; Spatial graphs

1. Introduction

By a graph we mean a finite undirected graph (loops and multiple edges allowed). A graph is *complete* if every pair of vertices is adjacent. A complete graph with n vertices is denoted by $\mathbf{K}_n$. A graph is said to be *bipartite* if its set of vertices can be partitioned into two disjoint sets A and B (bipartition) such that no two vertices in the same set are adjacent. A *complete* bipartite graph is a bipartite graph with bipartition A and B in which each vertex of A is joined to each vertex in B. A complete bipartite graph with $|A| = m$ and $|B| = n$ is denoted by $\mathbf{K}_{m,n}$. A *cycle* in a graph G is a connected subgraph G' of G such that every vertex of G' has exactly two incident edges. In particular, a cycle that contains every vertex of G is called a *Hamiltonian* cycle of G; see [8] for further graph theory details.

A *spatial representation* $\mathscr{R}(G)$, of a graph G, is the embedded image of G in $\mathbb{R}^3$, that is, the vertices of G are distinct points in 3-dimensional space and the edges are simple

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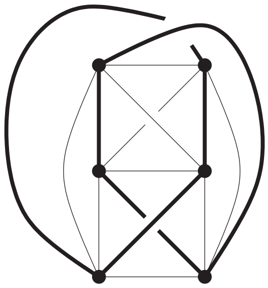

Fig. 1. A $\mathscr{R}(\mathbf{K}_6)$ with two concatenated cycles.

Jordan curves between them in such a way that any two curves are either disjoint or meet at a common end point. Throughout this paper we consider *tame* spatial representations, i.e., spatial representations that have piecewise linear edges. The cycles in $\mathscr{R}(G)$ can be thought of as simple closed curves in $\mathbb{R}^3$. Hence, we may regard a set of spatial cycles as a *link* in $\mathbb{R}^3$.

The study of the dependence (and independence) of the link (and knot) types contained in spatial representations has been the attention of recent research. These investigations have generated a number of research papers and, at present, there are not comprehensive or accessible sources summarizing the progress on it. This manuscript has a goal to survey what is known today. In Section 2 we describe and summarize the results in relation with *concatenated* graphs. Section 3 discusses the results on *self-knotted* graphs. Section 4 is dedicated to *realizable* embeddings. In Section 5 we survey what is known on *linear* embeddings. Finally, we give an Appendix with some basic definitions on knot theory needed throughout this paper.

2. Concatenated graphs

In 1973, Bothe [9] raised the following question:[1]

> *Consider a set of six distinct points in the Euclidean* 3*-dimensional space* $\mathbb{R}^3$ *and assume that each pair of these points is connected by a simple curve such that no two of these curves have a point in common—except for their endpoints, of course. Is it true that in this spatial figure we can always find a pair of simple closed curves that are* '*concatenated*'?

Two curves are called *concatenated* if they cannot be embedded in disjoint (topological) balls. Fig. 1 shows such spatial figure with exactly one pair of concatenated triangles.

Sachs [59] answered the above question positively (this was also answered by Conway and Gordon [16]).

[1] At the Eger Conference, W. Moser informed H. Sachs that the same problem had also been raised by his late brother Leo Moser about 15 years earlier (unpublished).