Available online at www.sciencedirect.com

SCIENCE DIRECT®

Discrete Mathematics 302 (2005) 243–253

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

On the crossing numbers of loop networks and generalized Petersen graphs

Gelasio Salazar[1]

Instituto de Fisica, UASLP, Alvaro Obregon 64 San Luis Potosi, SLP, Mexico 78000

Received 14 October 2002; received in revised form 13 December 2002; accepted 22 July 2004
Available online 10 August 2005

Abstract

Bhatt and Leighton proved that the crossing number of a network (graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for that network. With this important application in mind, it makes most sense to analyze the crossing numbers of graphs with good interconnection properties, such as the circulant graphs $G(n; \pm s_1, \ldots, \pm s_m)$. In this work we find tight bounds for the crossing numbers of the (double fixed step) circulant graphs $G(n; \pm 1, \pm k)$. Specifically, we show that for values of n sufficiently large compared to k, the crossing number of $G(n; \pm 1, \pm k)$ is bounded by above and by below by linear functions of n, both of which have coefficients that approach 1 as k goes to infinity. As an additional application of these bounds, we show that the crossing numbers of the generalized Petersen graphs $P(n, k)$ are bounded by below and by above by linear functions of n, whose coefficients approach $\frac{2}{5}$ and 2, respectively, as k goes to infinity.

MSC: 05C10; 05C62

Keywords: Crossing numbers; Circulant; Loop networks; Interconnection networks; VLSI layout; Generalized Petersen graphs

1. Introduction

One of the major motivations for the study of crossing numbers of graphs arises from the relationship between the crossing number of a network and the layout area required for the fabrication of a VLSI circuit for that network [4,18,24,25].

[1] Supported by CONACYT (Grant J32168E) and by FAI-UASLP.

E-mail address: gsalazar@ifisica.uaslp.mx.

0012-365X/$ - see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.disc.2004.07.036

From this perspective, the analysis of crossing numbers of graphs makes most sense when one focuses on graphs that have good properties as interconnection networks, such as meshes of trees [5–7] or circulant graphs. The (undirected) *circulant graph* $G(n; \pm s_1, \pm s_2, \ldots, \pm s_m)$ has n vertices $v(0), v(1), \ldots, v(n-1)$, so that $v(i)$ is adjacent to the $2m$ vertices $v(i \pm s_1), v(i \pm s_2), \ldots, v(i \pm s_m)$ (indices are read modulo n). In this paper we focus on *double fixed step graphs* $G(n; \pm s_1, \pm s_2)$, and in particular on graphs of the type $G(n; \pm 1, \pm k)$ (see [3,8,15,16,26]). The comprehensive survey [2] is a standard reference for circulant graphs (and more general distributed loop networks).

See Fig. 1 for an embedding of $G(n; \pm 1, \pm k)$ in the torus.

In this paper we prove that, for fixed k, the crossing number of $G(n; \pm 1, \pm k)$ is tightly bounded by above and by below by linear functions of n, whose coefficients approach 1 as k goes to infinity.

Theorem 1. *Let k, n be integers such that $k \geqslant 5$ and $n \geqslant k^4$. Then*

$$\left(1 - \frac{4}{k}\right) n + (4k^2 + 1 - k^3) \leqslant \mathrm{cr}(G(n; \pm 1, \pm k)) \leqslant \left(1 - \frac{2}{k}\right) n + (k^2/2 + k/2 + 1).$$

We also give an argument to show that these bounds imply the following bounds for the crossing numbers of generalized Petersen graphs $P(n, k)$ (see definition below).

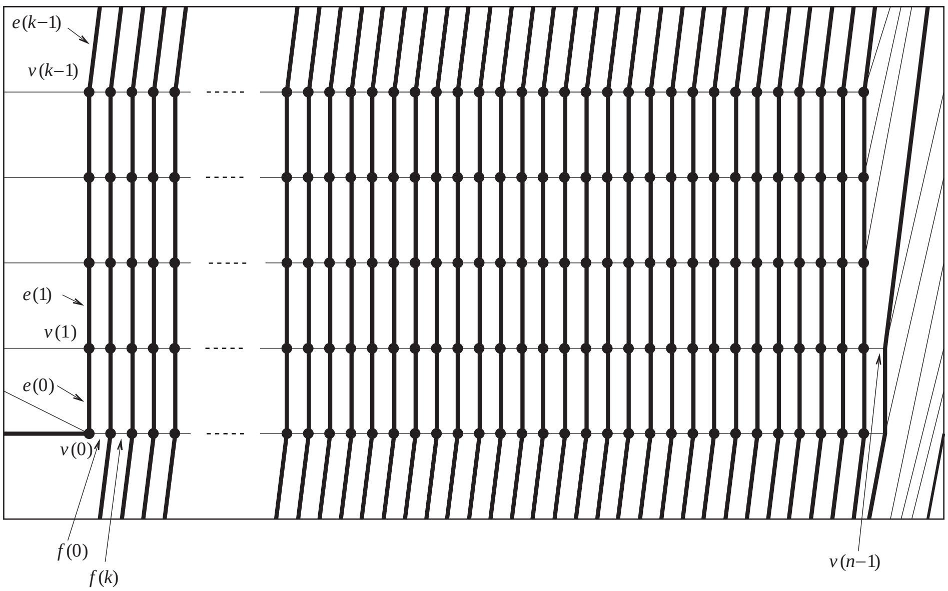

Fig. 1. An embedding of $G(n; \pm 1, \pm k)$ in the torus (here $k = 5$, and $n \bmod k = 2$). The edges $e(i)$ are drawn with thick segments, and the edges $f(i)$ are drawn with thin segments. This is easily generalized to an embedding of $G(n; \pm 1, \pm k)$ for arbitrary n and k.