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Simple permutations and pattern restricted permutations

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Abstract

A simple permutation is one that does not map any non-trivial interval onto an interval. It is shown that, if the number of simple permutations in a pattern restricted class of permutations is finite, the class has an algebraic generating function and is defined by a finite set of restrictions. Some partial results on classes with an infinite number of simple permutations are given. Examples of results obtainable by the same techniques are given; in particular it is shown that every pattern restricted class properly contained in the 132-avoiding permutations has a rational generating function.

Keywords: Permutation; Enumeration; Pattern-restricted

1. Introduction and definitions

In [14] Simion and Schmidt managed to enumerate the number of permutations of each length that avoided some arbitrary given set of permutation patterns of length 3. Their paper began the systematic study by many authors [2,5–8,10,11,15] of sets of permutations characterised by a set of avoidance conditions. The techniques in these papers tend to be tailored to the particular avoidance conditions at hand and very little in terms of a general theory has yet emerged. In this paper we shall go some way towards developing a general strategy for carrying out enumeration, and for answering other structural questions about restricted permutations.

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The principal tool in our work is the notion of a simple permutation (defined below). We shall show that a knowledge of the simple permutations in a pattern restricted class is often the key to understanding enough of its structure to carry out an enumeration and to answer the related question of whether a finite number of restrictions suffices to define the class. Our results completely answer both these questions when the number of simple permutations in the class is finite but they also have implications in more general cases.

Our paper is laid out as follows. The remainder of this section defines the necessary terminology including the definition of a simple permutation. Then, in Section 2, we explain how arbitrary permutations are built from simple ones and how this impacts on the minimal restrictions of a pattern closed class. Section 3 gives a key property of simple permutations that we exploit in the following section when discussing the number of restrictions. The core section is Section 5. There we show that the hypothesis of a finite number of simple permutations enables one to solve the enumeration problem (in theory and in practice). Section 6 gives some examples of how our techniques can be applied and we conclude with an overview and some unsolved problems.

A permutation π is a bijective function from $[n] = \{1, 2, \ldots, n\}$ to $[n]$ for some natural number n which is called the degree, or sometimes length, of π. To specify a permutation explicitly we usually write down the sequence of its values. Sets of permutations are denoted by calligraphic letters, $\mathscr{A}$, $\mathscr{B}$ etc. The set of all permutations is denoted $\mathscr{S}$, and $\mathscr{S}_n$ denotes the set of all permutations of length n. If $\mathscr{A}$ is a set of permutations, then A is the ordinary generating function for $\mathscr{A}$, that is

$$A(x) = \sum_{n=1}^{\infty} |\mathscr{A} \cap \mathscr{S}_n| x^n.$$

The *involvement* (sometimes called pattern containment) relation on $\mathscr{S}$ is a partial order $\preccurlyeq$ on $\mathscr{S}$ defined as follows: $\alpha \preccurlyeq \beta$ if and only if there is a subsequence of the sequence of values of β whose relative ordering is the same as the sequence of all values of α. Thus $231 \preccurlyeq 31524$ because the latter contains the subsequence 352 whose relative ordering is the same as that in 231. The relative ordering of a sequence will sometimes be called its *pattern*. Thus, any finite sequence without repetitions from a linearly ordered set has a unique pattern which is a permutation of the same length.

A *pattern class*, or simply *class*, is a collection of permutations closed downwards under $\preccurlyeq$. If $\mathscr{A}$ is a class and $\pi \notin \mathscr{A}$, then no element of $\mathscr{A}$ involves π. In this case we say that π is a *restriction* of $\mathscr{A}$. If in addition π is minimal with respect to $\preccurlyeq$ among the restrictions of $\mathscr{A}$, then we say that π is a *basic restriction* of $\mathscr{A}$. The set of basic restrictions of $\mathscr{A}$ is called the *basis* of $\mathscr{A}$ and denoted **basis**(A). Thus we have:

$$\mathscr{A} = \bigcap_{\pi \in \mathbf{basis}(\mathscr{A})} \{\theta : \pi \not\preccurlyeq \theta\}.$$

If $\mathscr{C}$ is any set of permutations and $\tau_1, \tau_2, \ldots, \tau_k$ are permutations, then we denote the subset of $\mathscr{C}$ consisting of those permutations involving none of $\tau_1, \tau_2, \ldots, \tau_k$ by