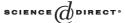


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Note

An eigenvalue bound for the Laplacian of a graph

J.P. Grossman

Department of Mathematics and Statistics, University of Dalhousie, Halifax, Canada B3H 3J5

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Abstract

We present a lower bound for the smallest non-zero eigenvalue of the Laplacian of an undirected graph. The bound is primarily useful for graphs with small diameter. © 2005 Elsevier B.V. All rights reserved.

Keywords: Graph Laplacian; Graph eigenvalues; Eigenvalue bounds

1. Introduction

We use the following definition for the Laplacian matrix of a graph, consistent with [3]:

Definition. Let G be an undirected graph with adjacency matrix \mathbf{A} , and let \mathbf{D} be the diagonal degree matrix defined by $d_{ii} = \deg(v_i)$ and $d_{ik} = 0$ for $i \neq k$. The *Laplacian* of G is the matrix $\mathcal{L} = \mathbf{I} - \mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1/2}$.

The eigenvalues of \mathcal{L} are in the range [0, 2]. Zero is always an eigenvalue with multiplicity equal to the number of connected components of G, and 2 occurs as an eigenvalue if and only if G is bipartite. The eigenvalues of \mathcal{L} contain additional information regarding the structure of G. They can be used to establish bounds on the diameter of G as well as distances between subsets of G [1,4,2,5]. The magnitudes of the eigenvalues also determine the rate of convergence of various iterative computations such as those described in [6,7]; it is therefore desirable to find bounds on the eigenvalues themselves. One of the best lower bounds for

 $\hbox{\it E-mail address:} jpg@alum.mit.edu.$

the smallest non-zero eigenvalue λ_1 is established in [3]

$$\lambda_1 \geqslant \frac{1}{D \operatorname{vol}(G)}.$$
 (1)

Here D is the diameter of G, and vol(G) is the sum of the degrees of all vertices. In this paper, we present a lower bound on λ_1 which is easy to compute and is tighter than (1) for certain graphs with low diameter.

2. A lower bound for λ_1

Consider the similar matrix $\mathbf{L} = \mathbf{D}^{-1/2} \mathcal{L} \mathbf{D}^{1/2} = \mathbf{I} - \mathbf{D}^{-1} \mathbf{A}$ which has the same eigenvalues as \mathcal{L} . If f is an eigenfunction of \mathbf{L} corresponding to eigenvalue λ then for any vertex v

$$(1 - \lambda)f(v) = \frac{1}{\deg(v)} \sum_{u \ge v} f(u),\tag{2}$$

where $u \sim v$ denotes that the vertices u, v are connected. Let $v_1, v_2, \ldots, v_{m+1}$ be a sequence of connected vertices such that $f(v_1)$ is maximal and $f(v_{m+1}) \leq 0$. For convenience, set $x_i = f(v_i)$. Let $\alpha = 1 - \lambda$, and let d be the maximum degree of any vertex of G. Since $v_1 \sim v_2$ and $f(v_1)$ is maximal, Eq. (2) gives us

$$\alpha x_1 = \frac{1}{\deg(v_1)} \sum_{u \sim v_1} f(u) \leqslant \frac{x_2}{\deg(v_1)} + \frac{(\deg(v_1) - 1)x_1}{\deg(v_1)} \leqslant \frac{x_2}{d} + \frac{(d - 1)x_1}{d}.$$
 (3)

Similarly, since $v_i \sim v_{i-1}$ and $v_i \sim v_{i+1}$ for $2 \le i \le m$, we have

$$\alpha x_i \leqslant \frac{x_{i-1} + x_{i+1}}{d} + \frac{(d-2)x_1}{d}.$$
 (4)

Scaling f if necessary, we may assume that $x_1 = 1$ and rewrite inequalities (3) and (4) as

$$x_2 \ge 1 - \lambda d,$$

 $x_{i+1} \ge \alpha dx_i - x_{i-1} - (d-2).$ (5)

We assume that $\lambda < 1$ and $\alpha d \ge 1$; otherwise we have the bound $\lambda > (d-1)/d$ which is much better than the one we will derive.

Lemma. For $3 \le k \le m+1$ we have $x_k \ge 1 - \lambda \alpha^{k-3} d^{k-2} - \lambda \alpha^{k-2} d^{k-1}$.

Proof. Our proof is by induction. Setting i = 2 in inequality (5) establishes the base case:

$$x_3 \ge \alpha dx_2 - 1 - (d - 2) \ge \alpha d(1 - \lambda d) + 1 - d = 1 - \lambda d - \lambda \alpha d^2$$
.

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