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Note

Various inverses of a strong endomorphism of a graph

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Abstract

In this paper, we first give a characterization of a completely regular strong endomorphism of a graph. Then we explicitly exhibit its various inverses. The enumerations of them are also derived.

Keywords: Graph; Strong endomorphism; Inverse

1. Introduction

As one of the most important semigroups associated with a graph, the endomorphism monoid of a graph, in particular, the strong endomorphism monoid of a graph, has received attention in the literature in recent years. For a survey see [6,2]. The aim of the researches in this line lies in establishing the connection between semigroup theory and graph theory, especially, in the applications of the former to the latter.

For a concrete semigroup, it seems always significant to be concerned with taking abstract semigroup concepts and describing them in this concrete semigroup. My writing this paper is motivated by a problem posed to me by Professor T. E. Hall (Monash University, Australia): How can the inverses of a strong endomorphism of a graph be described?

It is well known that in the theory of semigroups, an idea of great importance is that of an inverse associated with a regular element [1, p. 45]. In [4], the inverses of a regular endomorphism of a graph were investigated. In [2,3], it was proved that for any finite graph G, $\mathrm{sEnd}(G)$, the monoid of the strong endomorphisms of G, is always regular. So, every

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strong endomorphism of *G* possesses pseudo-inverses as well as inverses. A concept stronger than regularity in the semigroup theory is that of complete regularity. Every completely regular element of a semigroup has a commuting inverse and, of course, a commuting pseudo-inverse [5, Theorem II.6.7].

In this paper, we first give a characterization of a completely regular strong endomorphism of a graph. Then, we explicitly exhibit the pseudo-inverses and inverses of a strong endomorphism of a graph, as well as commuting pseudo-inverses and commuting inverses of a completely regular strong endomorphism of a graph, by which the enumerations of them are also derived.

In this paper, only finite undirected graphs without loops and multiple edges are considered. If *G* is a graph, we denote by $V(G)$ (or just *G*) and $E(G)$ its vertex set and edge set, respectively. A graph *H* is called a *subgraph* of *G* if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(G)$, then we call *H* an *induced* (*strong*) *subgraph* of *G*. For graphs *G* and *H*, a mapping $f : V(G) \to V(H)$ is called a *homomorphism* from *G* to *H* if $\{a, b\} \in E(G)$ implies that $\{f(a), f(b)\} \in E(H)$ for any $a, b \in G$. Moreover, if *f* is bijective and its inverse mapping is also a homomorphism (from *H* to *G*), then *f* is called an *isomorphism* from *G* to *H*. An *endomorphism* of *G* is a homomorphism from *G* to itself. An endomorphism is called a *strong endomorphism* if $\{f(a), f(b)\} \in E(G)$ implies that $\{a, b\} \in E(G)$ for any $a, b \in G$. A bijective endomorphism of a graph *G* is called an *automorphism* of *G*. Evidently, an automorphism of a graph *G* is an isomorphism from *G* to itself. By $\mathrm{End}(G)$, $\mathrm{sEnd}(G)$ and $\mathrm{Aut}(G)$ denote the set of endomorphisms, strong endomorphisms and automorphisms of the graph *G*, respectively. Obviously, $\mathrm{Aut}(G) \subseteq \mathrm{sEnd}(G) \subseteq \mathrm{End}(G)$. It is well-known that $\mathrm{End}(G)$ and $\mathrm{sEnd}(G)$ are monoids (a *monoid* is a semigroup with an identity element) and that $\mathrm{Aut}(G)$ is a group with respect to the composition of mappings. We denote an endomorphism *f* in the obvious sense as, e.g., $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$. Let *G* be a graph and let $A \subseteq V(G)$. Let $f \in \mathrm{sEnd}(G)$ and let $a \in G$. We will denote $f^{-1}(a) := \{x \in G | f(x) = a\}$, $f(A) := \{f(x) | x \in A\}$ and $f^{-1}(A) := \bigcup_{x \in A} f^{-1}(x)$. By $f|_A$ we denote the restriction of *f* on *A*.

The following notions are important in this paper: Let $f \in \mathrm{sEnd}(G)$, and by S_f we denote the induced subgraph of *G* with $V(S_f) = f(V(G))$; by ρ_f we denote the relation on $V(G)$ induced by *f*, namely, for any $a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. For a vertex $a \in G$, we put $N(a) = \{x \in G | \{a, x\} \in E(G)\}$ (the *neighborhood* of *a* in *G*). The relation τ on $V(G)$ is defined by the rule that $(a, b) \in \tau$ if and only if $N(a) = N(b)$. Clearly, the relations τ and ρ_f are equivalence relations on $V(G)$. By $[x]_\tau$ (resp. $[x]_{\rho_f}$) we denote the equivalence class of the vertex *x* of graph *G* with respect to τ (resp. ρ_f).

Let *S* be a semigroup. An *idempotent* is an element *a* of *S* such that $a^2 = a$. An element *a* of *S* is said to be *regular* if there exists *x* in *S* such that $axa = a$. In this case, the element *x* is called a *pseudo-inverse* of *a* [2]. Furthermore, if $xax = x$ is also true, then *x* is called an *inverse* of *a* [1]. A semigroup is said to be *regular* if all its elements are regular [1]. An element *a* of a semigroup *S* is called *completely regular* if there exists an element *x* in *S* such that $axa = a$ and $xa = ax$ [1,5]. If this is the case, we call the element *x* a *commuting pseudo-inverse* of *a*. Moreover, if $xax = x$ also holds, then *x* is called a *commuting inverse* of *a*. Define a relation $\mathscr{L}$ on *S* such that $(a, b) \in \mathscr{L}$ if and only if $S^1 a = S^1 b$. Similarly, define a relation $\mathscr{R}$ on *S* such that $(a, b) \in \mathscr{R}$ if and only if $aS^1 = bS^1$. Define $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$.