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# Association schemes based on isotropic subspaces, Part 1

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**Abstract**

The subspaces of a given dimension in a finite classical polar space form the points of an association scheme. When the dimension is zero, this is the scheme of the collinearity graph of the space. At the other extreme, when the dimension is maximal, it is the scheme of the corresponding dual polar graph. These extreme cases have been thoroughly studied. In this article, the general case is examined and a detailed computation of the intersection numbers of these association schemes is initiated.


*Keywords:* Association scheme; Intersection number; Polar space; Finite field; Isotropic subspace

## 1. Introduction

Grassman graphs and dual polar graphs are two well-known classes of distance-regular graphs. The intersection numbers and eigenvalues of these and other distance-regular graphs with "classical parameters" are recorded in [4, Chapter 9]. In the case of the dual polar graphs, they were determined by D. Stanton in [14]. In his Ph.D. dissertation [10], the author of the present article used a number of formulas from [7] to give an alternative derivation of the intersection numbers found in [14]. Also in [10], adjacency in a graph which the author calls a *hyperbolic partner graph* was observed to be part of an association scheme, although this fact is an easy consequence of Witt's Extension Theorem. Actually, there are classes of such graphs and related association schemes. These association schemes include those of the dual polar graphs as well as the collinearity graphs of finite classical polar spaces, and

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are rather evident generalizations of these. They can also be viewed as a type of analogue of the Grassman graph association schemes, which in turn are "$q$-analogues" of Johnson graph association schemes.

The association schemes to be considered are defined as follows. Fix an $N$-dimensional vector space $V$ over a finite field GF($q$). In order to simplify the discussion, it will be assumed that $q$ is odd. Equip this space with a non-degenerate form $\langle\cdot,\cdot\rangle$ that is symmetric bilinear, alternating bilinear or Hermitian. Let $d$ be the Witt index of this form (i.e. the dimension of any maximal isotropic subspace), and assume that $d \geqslant 1$. Fix an integer $m$ between 1 and $d$. The points of the association scheme will be the isotropic (i.e. totally singular) $m$-subspaces of $V$.

If $U$ and $U'$ are isotropic $m$-subspaces of $V$, and if $\dim(U \cap U') = m - k$, and $\dim(U^{\perp} \cap U') = m - \gamma$, then we will we say that these two subspaces are $(k, \gamma)$-*associates*. $U^{\perp}$ is defined to be $\{v \in V \mid (\forall u \in U)\langle u, v\rangle = 0\}$. Clearly $0 \leqslant \gamma \leqslant k \leqslant m$ here since $U \subseteq U^{\perp}$. It is not hard to see that $\dim(U \cap U'^{\perp})$ will also equal $m - \gamma$ as follows. $(U \cap U'^{\perp})^{\perp} = U^{\perp} + U'$ has dimension $(N - m) + m - (m - \gamma) = N - (m - \gamma)$, so $U \cap U'^{\perp}$ has dimension $m - \gamma$. The ordered pair $(k, \gamma)$ will be written using the following unorthodox notation: $k_\gamma$. Thus for example, $5_2$ really means (5,2). We will speak of $k_\gamma$-associates, rather than $(k, \gamma)$-associates. Two isotropic subspaces that are $k_\gamma$-associates are a distance $k$ apart in the Grassman graph whose vertices are the $m$-subspaces of $V$.

Let $\mathscr{N}_m$ denote the collection of all isotropic $m$-subspaces of $V$. Let $\mathscr{R}_{m,k_\gamma}$ be the collection of all ordered pairs of isotropic $m$-subspaces of $V$ that are $k_\gamma$-associates. If $m$ is implicit, then $\mathscr{R}_{k_\gamma}$ will be written instead of $\mathscr{R}_{m,k_\gamma}$. The claim of course is that with $\mathscr{N}_m$ as the set of "points", the relations $\mathscr{R}_{k_\gamma}$ $(0 \leqslant \gamma \leqslant k \leqslant m)$ form an association scheme. It is an immediate consequence of Witt's Extension Theorem that when the isometry group of form $\langle\cdot,\cdot\rangle$ acts (in the canonical way) on $\mathscr{N}_m$, this action is generously transitive and the orbitals are in fact the relations $\mathscr{R}_{k_\gamma}$. It follows that these relations do indeed form an association scheme. This is the same reasoning typically used to establish that Grassman graphs and dual polar graphs are distance-transitive (cf. [4, Theorems 9.3.1, 9.3.3, 9.4.3]).

There are actually six distinct types of forms on $V$ ("geometries") to consider. If the form is alternating, then $V$ is said to have a *symplectic geometry* or a geometry of type $C_d$. Here $N = 2d$. If the form is symmetric, then one speaks of $V$ having an *orthogonal geometry*. Here if $n$ is odd, then $n$ will equal $2d + 1$ and the geometry is said to be of type $B_d$. But if $n$ is even, then it is possible that either $N = 2d$ or $N = 2d + 2$, and one speaks of geometries of type $D_d$ or ${}^2D_{d+1}$, accordingly. Finally, if the form is Hermitian, then $V$ is said to have a *unitary geometry*. If $N$ is even, then $N = 2d$, and the geometry is said to be of type ${}^2A_{2d-1}$, while if $N$ is odd, then $N = 2d + 1$, and the geometry is said to be of type ${}^2A_{2d}$. Following notation introduced in [10], let $\mu = \frac{1}{2}N - d$ and let $v$ be such that $\mu + v$ equals 0, $\frac{1}{2}$, 1 in the symplectic, unitary, orthogonal cases, respectively. The parameters for the six types of geometries are shown in Fig. 1.

We will require some additional terminology and notation. $\mathscr{L}(V)$ will denote the lattice of all subspaces of $V$. Given $U \in \mathscr{L}(V)$ (i.e. a subspace of $V$), the *radical* of $U$, denoted rad($U$), is simply the subspace $U \cap U^{\perp}$, which is necessarily isotropic. Note that $U$ is isotropic if and only if $U = \mathrm{rad}(U)$ $(U \subseteq U^{\perp})$. When $U$ is isotropic, the quotient space $U^{\perp}/U$ inherits the form on $V$ by defining $\langle u + U, v + U\rangle = \langle u, v\rangle$, and this inherited form is non-degenerate and produces the same type of geometry on $U^{\perp}/U$ as on $V$. Also,