Available online at www.sciencedirect.com
SCIENCE DIRECT

Discrete Mathematics 297 (2005) 159–166

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

Reconstructing subgraph-counting graph polynomials of increasing families of graphs☆

Boštjan Brešar[a], Wilfried Imrich[b], Sandi Klavžar[c]

[a] University of Maribor, FERI, Smetanova 17, 2000 Maribor, Slovenia
[b] Montanuniversität Leoben, A-8700 Leoben, Austria
[c] Department of Mathematics and Computer Science, University of Maribor, PeF, Koroška cesta 160, 2000 Maribor, Slovenia

Received 11 September 2002; received in revised form 26 October 2003; accepted 17 February 2005
Available online 18 July 2005

Abstract

A graph polynomial $P(G, x)$ is called reconstructible if it is uniquely determined by the polynomials of the vertex-deleted subgraphs of G for every graph G with at least three vertices. In this note it is shown that subgraph-counting graph polynomials of increasing families of graphs are reconstructible if and only if each graph from the corresponding defining family is reconstructible from its polynomial deck. In particular, we prove that the cube polynomial is reconstructible. Other reconstructible polynomials are the clique, the path and the independence polynomials. Along the way two new characterizations of hypercubes are obtained.

Keywords: Graph polynomial; Hypercube; Reconstruction; Cube polynomial; Increasing family of graphs

1. Introduction

Let G be a simple graph with a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and let $G_i = G - v_i$, $1 \leqslant i \leqslant n$, be its vertex-deleted subgraph. Then, the multiset $\{G_1, G_2, \ldots, G_n\}$ is called the *deck* of G. A graph G is called *reconstructible* if it is uniquely determined (up to

☆ Supported by the Ministry of Science of Slovenia under the grants Z1-3073-0101-01 and 0101-P-504, and by the Österreichisches Ost- und Südosteuropa Institut.

E-mail addresses: bostjan.bresar@uni-mb.si (B. Brešar), imrich@unileoben.ac.at (W. Imrich), sandi.klavzar@uni-mb.si (S. Klavžar).

doi:10.1016/j.disc.2005.02.019

isomorphism) by its deck. The well-known *reconstruction conjecture* (also known as the Kelly–Ulam conjecture) asserts that all finite graphs on at least three vertices are reconstructible, cf. [1].

More generally, a *graph property is reconstructible* if it is uniquely determined by the deck of a graph. Many graph properties have been proved to be reconstructible, for instance, the number of Hamiltonian cycles and the number of one-factors, cf. [18] for these and many more properties. In addition, Tutte [26] proved that the characteristic polynomial, the chromatic polynomial, and their generalizations are also reconstructible, in addition to the matching polynomial [9,13]. For more information on the reconstruction of classical graph polynomials see the survey [7]. In the same paper Farrell also observed that the reconstruction conjecture can be stated in terms of reconstructible graph polynomials.

Given a graph property, do we really need the deck of a graph for its reconstruction? In particular, given a graph polynomial, can it be reconstructed from its *polynomial deck*, that is, from the multiset of the polynomials of the vertex-deleted subgraphs? For the characteristic polynomial, Gutman and Cvetković [15] posed this question in 1975, but the problem remains open. Recently, Hagos [16] proved that the characteristic polynomial of a graph is reconstructible from the polynomial deck of a graph together with the polynomial deck of its complement. For related results see [25] and [24]; in the latter paper Schwenk supposes that the answer to the question is negative.

In this paper we consider the problem of reconstructing a graph polynomial from its polynomial deck for a class of polynomials that are defined as generating functions for the numbers of subgraphs from given increasing families of graphs. These subgraph-counting polynomials are instances of F-polynomials in the sense of Farrell [6].

In the next section we formally introduce these polynomials and prove that such polynomials are reconstructible from the polynomial deck if and only if each graph from the corresponding defining family is reconstructible from its polynomial deck. The well-known clique, independence, star and path polynomial as well as the recently introduced cube polynomial [3] are of this type . In Section 3 we prove that the cube polynomial is also reconstructible. Two related characterizations of hypercubes are given, for example, a graph is a hypercube if and only if its cube polynomial is of the form $(x+2)^k$.

2. Reconstruction of $\mathscr{H}$-polynomials

Let $\mathscr{H} = \{H_0, H_1, H_2, \ldots\}$ be a family of graphs such that $H_0 = K_1$ and H_i is an induced subgraph of H_{i+1} for $i = 0, 1, 2, \ldots$. We call such a family of graphs an *increasing family*. Given an increasing family $\mathscr{H}$, and an arbitrary graph G, we denote by $p_i(G)$ the number of induced H_i's in G. The *$\mathscr{H}$-polynomial* $P_{\mathscr{H}}(G, x)$ of a graph G is the generating function for the $p_i(G)$, that is,

$$P_{\mathscr{H}}(G, x) = \sum_{i \geqslant 0} p_i(G) x^i. \tag{1}$$

For example, setting $H_i = K_i$, respectively, $H_i = \overline{K_i}$ or $H_i = Q_i$, one obtains the *clique polynomial* [10,17], the *independence polynomial* [4,14,17], and the *cube polynomial* [3].