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On average lower independence and domination numbers in graphs

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Abstract

The average lower independence number $i_{\text{av}}(G)$ of a graph $G=(V,E)$ is defined as $\frac{1}{|V|}\sum_{v\in V} i_v(G)$, and the average lower domination number $\gamma_{\text{av}}(G)$ is defined as $\frac{1}{|V|}\sum_{v\in V}\gamma_v(G)$, where $i_v(G)$ (resp. $\gamma_v(G)$) is the minimum cardinality of a maximal independent set (resp. dominating set) that contains v. We give an upper bound of $i_{\text{av}}(G)$ and $\gamma_{\text{av}}(G)$ for arbitrary graphs. Then we characterize the graphs achieving this upper bound for i_{av} and for γ_{av} respectively.

MSC: 05C69

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1. Introduction

In a graph $G=(V(G),E(G))$, a subset $S\subseteq V$ of vertices is a *dominating set* if every vertex in $V(G)-S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. The *independent domination number* $i(G)$ is the minimum cardinality of a set that is both independent and dominating. The independence number $\alpha(G)$ is the maximum cardinality of an independent set. It is easy

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to see that $\gamma(G) \leqslant i(G) \leqslant \alpha(G)$ holds for every graph G. For a comprehensive treatment of domination in graphs, see [6,7].

Henning [8] introduced the concept of average independence and average domination. For a vertex v of a graph G, the *lower independence number*, denoted by $i_v(G)$, is the minimum cardinality of a maximal independent set of G that contains v, and the *lower domination number*, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of G that contains v. It is easy to see that every maximal independent set is a dominating set, and so $\gamma_v(G) \leqslant i_v(G)$ holds for every vertex v. The *average lower independence number* of G, denoted by $i_{\mathrm{av}}(G)$, is the value $\frac{1}{|V(G)|}\sum_{v\in V(G)} i_v(G)$, and the *average lower domination number* of G, denoted by $\gamma_{\mathrm{av}}(G)$, is the value $\frac{1}{|V(G)|}\sum_{v\in V(G)} \gamma_v(G)$. Since $\gamma_v(G) \leqslant i_v(G)$ holds for every vertex v, we have $\gamma_{\mathrm{av}}(G) \leqslant i_{\mathrm{av}}(G)$ for any graph G. Also, it is clear that $i(G) = \min\{i_v(G) \mid v \in V(G)\}$, $\gamma(G) = \min\{\gamma_v(G) \mid v \in V(G)\}$ and so $\gamma(G) \leqslant \gamma_{\mathrm{av}}(G)$, and $i(G) \leqslant i_{\mathrm{av}}(G)$.

Henning [8] established an upper bound for the average lower independence number of a tree and characterized the trees that achieve equality for this bound.

Theorem 1 (*Henning [8]*). *If T is a tree of order $n \geqslant 2$, then*

$$i_{\mathrm{av}}(T) \leqslant n - 2 + \frac{2}{n}, \tag{1}$$

with equality if and only if T is a star $K_{1,n-1}$.

In this paper, we give an upper bound for the average lower independence and domination numbers for any graph, improving Henning's bound for trees. Then we characterize the graphs attaining this upper bound for the average lower independence and domination numbers respectively.

We finish this section by recalling some terminology and notation. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex v of G, the *open neighbourhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. Also we write $\overline{N}(v) = V(G) - N[v]$. If $S \subseteq V(G)$ then $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$. The *degree* of a vertex v of G, denoted by $d(v)$, is the size of its open neighbourhood. A vertex v of degree 1 (resp. degree 0) is called a *pendant vertex* (resp. an *isolated vertex*). We denote by n the *order* of G, which is the size of $V(G)$. For a subset A of $V(G)$, $G[A]$ will denote the subgraph induced by the vertices of A.

2. Upper bound

A *matching* in a graph G is a subset of pairwise non-incident edges. The *matching number* $\beta(G)$ is the size of a largest matching in G. A matching is said to be *perfect* if $\beta(G) = n/2$. For any vertex $v \in V(G)$, let $\beta_v(G)$ be the maximum cardinality of a matching in the graph induced by the vertices of $V(G) - N[v]$, that is, $\beta_v(G) = \beta(G[V(G) - N[v]])$. Recall that $\beta_v(G)$ can be computed for any graph G in polynomial time (see [4]). Blidia et al. [2] gave an upper bound for the lower domination parameters $i(G)$ and $\gamma(G)$ for any graph G.