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A vertex ordering characterization of simple-triangle graphs

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ABSTRACT

Consider two horizontal lines in the plane. A point on the top line and an interval on the bottom line define a triangle between two lines. The intersection graph of such triangles is called a simple-triangle graph. This paper shows a vertex ordering characterization of simple-triangle graphs as follows: a graph is a simple-triangle graph if and only if there is a linear ordering of the vertices that contains both an alternating orientation of the graph and a transitive orientation of the complement of the graph.

1. Introduction

Let L_1 and L_2 be two horizontal lines in the plane with L_1 above L_2. A pair of a point on the top line L_1 and an interval on the bottom line L_2 defines a triangle between L_1 and L_2. The point on L_1 is called the *apex* of the triangle, and the interval on L_2 is called the *base* of the triangle. A *simple-triangle graph* is the intersection graph of such triangles, that is, a simple undirected graph G is called a simple-triangle graph if there is such a triangle for each vertex and two vertices are adjacent if and only if the corresponding triangles have a nonempty intersection. The set of triangles is called a *representation* of G. See Figs. 1(a) and 1(b) for example. Simple-triangle graphs are also known as *PI graphs* [2,3,5], where *PI* stands for *Point-Interval*. Simple-triangle graphs were introduced as a generalization of both interval graphs and permutation graphs, and they form a proper subclass of trapezoid graphs [5]. Although a lot of research has been done for interval graphs, for permutation graphs, and for trapezoid graphs (see [2,10,12,17,19,23] for example), there are few results for simple-triangle graphs [1,3,5]. The polynomial-time recognition algorithm has been given recently [18,25], but the complexity of the graph isomorphism problem still remains an open question [24,26], which makes it interesting to study the structural characterizations of this graph class.

A *vertex ordering* of a graph $G = (V, E)$ is a linear ordering $\sigma = v_1, v_2, \dots, v_n$ of the vertex set V of G. We use $u <_\sigma v$ to denote that u precedes v in σ. A *vertex ordering characterization* of a graph class $\mathcal{G}$ is a characterization of the following type: a graph G is in $\mathcal{G}$ if and only if G has a vertex ordering fulfilling some properties. For example, it is known that a graph G is an interval graph if and only if G has a vertex ordering σ such that for any three vertices $u <_\sigma v <_\sigma w$, if $uw \in E$ then $uv \in E$ [21]. In other words, a graph is an interval graph if and only if it has a vertex ordering that contains no subordering in Figs. 2(a) and 2(c). It is also known that a graph is a permutation graph if and only if it has a vertex ordering that contains no subordering in Figs. 2(b) and 2(c). See [2,6,13] for other examples of vertex ordering characterizations.

This paper shows a vertex ordering characterization of simple-triangle graphs. We call a vertex ordering σ of a simple-triangle graph G an *apex ordering* if there is a representation of G such that σ coincides with the ordering of the apices of the triangles in the representation. See Fig. 1(d) for example. We characterize the apex orderings of simple-triangle graphs

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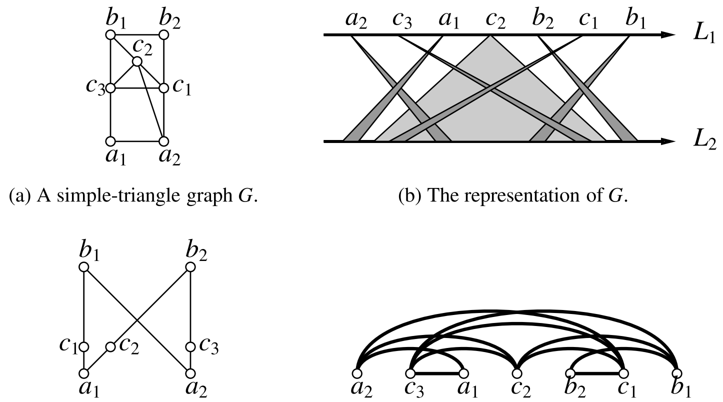


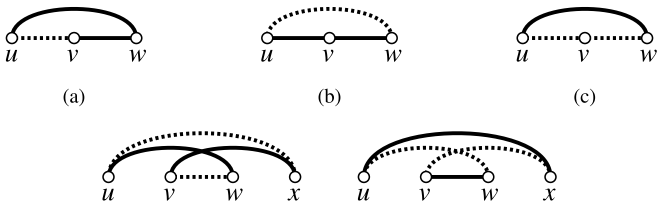
(a) A simple-triangle graph G.

(b) The representation of G.

(c) The linear-interval order P.

(d) The apex ordering of G.

Fig. 1. A simple-triangle graph G, the representation of G, the Hasse diagram of linear-interval order P, and the apex ordering of G.



(a) (b) (c) (d) (e)

Fig. 2. Forbidden patterns. Lines and dashed lines denote edges and non-edges, respectively. Edges that may or may not be present is not drawn.

as follows: a vertex ordering σ of a graph G is an apex ordering if and only if σ contains no subordering in Figs. 2(c)–2(e). Equivalently, we show that a vertex ordering σ of G is an apex ordering if and only if σ contains both an alternating orientation of G and a transitive orientation of the complement $\overline{G}$ of G.

The organization of this paper is as follows. Before describing the vertex ordering characterization, we show in Section 2 a characterization of the linear-interval orders, the partial orders associated with simple-triangle graphs. The vertex ordering characterization of simple-triangle graphs is shown in Section 3. We remark some open questions and related topics in Section 4.

2. Linear-interval orders

A *partial order* is a pair $P = (V, \prec_P)$, where V is a finite set and $\prec_P$ is a binary relation on V that is irreflexive and transitive. The finite set V is called the *ground set* of P. A partial order $P = (V, \prec_P)$ is called a *linear order* if for any two elements $u, v \in V$, $u \prec_P v$ or $u \succ_P v$. A partial order $P = (V, \prec_P)$ is called an *interval order* if for each element $v \in V$, there is a (closed) interval $I(v) = [l(v), r(v)]$ on the real line such that for any two elements $u, v \in V$, $u \prec_P v \iff r(u) < l(v)$, that is, $I(u)$ lies completely to the left of $I(v)$. The set of intervals $\{I(v) \mid v \in V\}$ is called an *interval representation* of P.

Let $P_1 = (V, \prec_1)$ and $P_2 = (V, \prec_2)$ be two partial orders with the same ground set. The *intersection* of P_1 and P_2 is the partial order $P = (V, \prec_P)$ such that $u \prec_P v \iff u \prec_1 v$ and $u \prec_2 v$; it is denoted by $P = P_1 \cap P_2$. A partial order P is called a *linear-interval order* (also known as a *PI order* [3]) if there exist a linear order L and an interval order P_I such that $P = L \cap P_I$. The linear-interval order can also be defined as follows. Recall that L_1 and L_2 are two horizontal lines with L_1 above L_2, and a pair of a point on L_1 and an interval on L_2 defines a triangle between L_1 and L_2. A partial order $P = (V, \prec_P)$ is a linear-interval order if there is such a triangle $T(v)$ for each element $v \in V$, and $u \prec_P v$ if and only if $T(u)$ lies completely to the left of $T(v)$. See Figs. 1(b) and 1(c) for example. Notice that the ordering of the apices of the triangles gives the linear order L, and the bases of the triangles give an interval representation of the interval order P_I.

A linear order $L = (V, \prec_L)$ is called a *linear extension* of a partial order $P = (V, \prec_P)$ if $u \prec_L v$ whenever $u \prec_P v$. Hence, the linear extension L of P has all the relations of P with the additional relations that make L linear. We define two properties of linear extensions.

– Let $\mathbf{2+2}$ denote the partial order consisting of four elements a_0, a_1, b_0, b_1 whose only relations are $a_0 \prec_P b_0$ and $a_1 \prec_P b_1$. A linear extension $L = (V, \prec_L)$ of $P = (V, \prec_P)$ is said to fulfill the $\mathbf{2+2}$ *rule* if for every suborder $\mathbf{2+2}$ in P, either $b_0 \prec_L a_1$ or $b_1 \prec_L a_0$.