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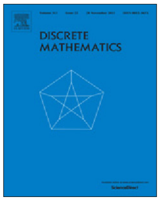

On partitions of $K_{2,3}$-free graphs under degree constraints[☆]

Jianfeng Hou [a], Huawen Ma [a,*], Jiguo Yu [b], Xia Zhang [c]

[a] *Center for Discrete Mathematics, Fuzhou University, Fujian, 350003, China*
[b] *School of Information Science and Engineering, Qufu Normal University, Shandong, 276826, China*
[c] *School of Mathematics and Statistics, Shandong Normal University, Shandong, 250014, China*

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ABSTRACT

Suppose that s, t are two positive integers, and $\mathscr{H}$ is a set of graphs. Let $g(s, t; \mathscr{H})$ be the least integer g such that any $\mathscr{H}$-free graph with minimum degree at least g can be partitioned into two sets which induced subgraphs have minimum degree at least s and t, respectively. For a given graph H, we simply write $g(s, t; H)$ for $g(s, t; \mathscr{H})$ when $\mathscr{H} = \{H\}$. In this paper, we show that if $s, t \geq 2$, then $g(s, t; K_{2,3}) \leq s + t$ and $g(s, t; \{K_3, C_8, K_{2,3}\}) \leq s + t - 1$. Moreover, if $\mathscr{H}$ is the set of graphs obtained by connecting a single vertex to exactly two vertices of $K_4 - e$, then $g(s, t; \mathscr{H}) \leq s + t$ on $\mathscr{H}$-free graphs with at least five vertices, which generalize a result of Liu and Xu (2017).

1. Introduction

Graph partitioning problems came from the well-known *Max-Cut Problem*: Given a graph G, find a maximum bipartite subgraph of G. A simple average argument shows that every graph with m edges has a bipartite subgraph with at least $m/2$ edges. In 1973, answering a question of Erdős, Edwards [4,5] improved this lower bound to $m/2 + (\sqrt{2m + 1/4} - 1/2)/4$, which is essentially best possible as evidenced by complete graphs with odd orders. For more references concerning partitions of graphs or hypergraphs, see [1,2,6–10,15,18,19].

Another direction of graph partitioning is to partition graphs under some degree constraints. An example given by Lovász [13] in 1966, shows us that every graph with maximum degree at most $s + t + 1$ admits a bipartition which induced subgraphs have maximum degree at most s and t, respectively. For graphs partitions under minimum degree constraints, Stiebitz [16] proved that every graph with minimum degree at least $s + t + 1$ admits a bipartition which induced subgraphs have minimum degree at least s and t, respectively, which confirmed a conjecture posed by Thomassen [17]. The complete graph K_{s+t+1} shows that the bound is sharp.

Suppose that $\mathscr{H}$ is a set of graphs. We say that a graph is $\mathscr{H}$-free if it contains no member of $\mathscr{H}$ as subgraphs. Let $g(s, t; \mathscr{H})$ be the least integer g such that any $\mathscr{H}$-free graph with minimum degree at least g can be partitioned into two sets which induced subgraphs have minimum degree at least s and t, respectively. For a given graph H, we simply write $g(s, t; H)$ for $g(s, t; \mathscr{H})$ when $\mathscr{H} = \{H\}$. There are reasonable bounds for $g(s, t; \mathscr{H})$. Kaneko [11] first studied $g(s, t; \mathscr{H})$ and showed that $g(s, t; K_3) \leq s + t$. Suppose that $s, t \geq 2$. Diwan [3] showed that $g(s, t; \{K_3, K_{2,2}\}) \leq s + t - 1$, which was improved to $g(s, t; K_{2,2}) \leq s + t - 1$ by Ma and Yang [14]. Our first result shows that $g(s, t; K_{2,3}) \leq s + t$.

In fact, the results mentioned above concern *feasible partition*. First, we describe notation and terminology. Let G be a graph, and let A, B be two nonempty subsets of $V(G)$. Then (A, B) is a *partition* of G if $A \cup B = V(G)$ and $A \cap B = \emptyset$. For $x \in V(G)$,

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* Corresponding author.
E-mail address: mhwaue@sina.com (H. Ma).

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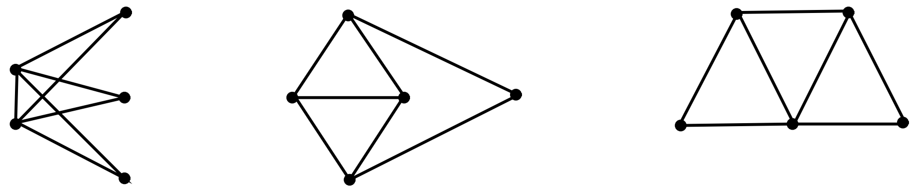

Fig. 1. From left to right, the graphs are $K_{2,3}^{+}$, H_1, H_2.

let $d_A(x)$ denote the number of vertices in A that are adjacent to x in G, and let $e(A)$ denote the number of edges with both ends in A. Let $a, b : V(G) \longmapsto \mathbf{N}$ be two functions. We say (A, B) is an (a, b)-*feasible partition* if $d_A(x) \geq a(x)$ for every $x \in A$ and $d_B(y) \geq b(y)$ for every $y \in B$. The following theorem gives an (a, b)-*feasible partition* of $K_{2,3}$-free graphs.

Theorem 1.1. *Let G be a $K_{2,3}$-free graph, and let $a, b : V(G) \longmapsto \mathbf{N} \setminus \{0, 1\}$ be two functions. If $d(x) \geq a(x) + b(x)$ for each $x \in V(G)$, then G admits an (a, b)-feasible partition.*

Remark that the icosahedron is 5-regular and $K_{2,3}$-free, but it does not admit (4,1)-feasible partitions. So the ranges of the functions a, b cannot be relaxed to the set of integers at least one. If we further forbid K_3 and C_8, then we can lower the degree condition further by the following theorem.

Theorem 1.2. *Let G be a $\{K_3, C_8, K_{2,3}\}$-free graph, and let $a, b : V(G) \longmapsto \mathbf{N} \setminus \{0, 1\}$ be two functions. If $d(x) \geq a(x) + b(x) - 1$ for each $x \in V(G)$, then G admits an (a, b)-feasible partition.*

Let K_4^- be the graph obtained from K_4 by deleting an edge. Recently, Liu and Xu [12] proved that if a graph G is a K_4^--free graph with at least four vertices and $d(x) \geq a(x)+b(x)$ for each vertex $x \in V(G)$, where $a, b : V(G) \longmapsto \mathbf{N} \setminus \{0\}$, then G admits an (a, b)-feasible partition. Let $\mathscr{H}$ be the set of graphs obtained by connecting a single vertex to exactly two vertices of K_4^- (see Fig. 1). It is easy to see that K_4^- is a subgraph of each graph in $\mathscr{H}$. The following theorem generalizes Liu and Xu's result.

Theorem 1.3. *Let G be a $\{K_{2,3}^{+}, H_1, H_2\}$-free graph with at least five vertices, and let $a, b : V(G) \longmapsto \mathbf{N} \setminus \{0\}$ be two functions. If $d(x) \geq a(x) + b(x)$ for each $x \in V(G)$, then G admits an (a, b)-feasible partition.*

Let $G = K_{2,3}^{+}$, and $a, b : V(G) \longmapsto \mathbf{N} \setminus \{0\}$ be two functions such that $a(x) = d(x) - 1$ and $b(x) = 1$ for each $x \in V(G)$. It is easy to see that G has no (a, b)-feasible partition. This implies that the condition $K_{2,3}^{+}$-free is necessary. By the similar arguments, we see that the conditions H_1-free and H_2-free are also necessary.

This paper is organized as follows. In Section 2, we introduce some notations and a key lemma used in our proofs. We prove Theorems 1.1, 1.2 and 1.3 in Sections 3, 4 and 5, respectively.

2. Preliminary

In this section, we introduce some notations and lemmas that are used in our proofs. A *k-cycle*, denoted by C_k, is a cycle of length k. Let G be a graph and $a, b : V(G) \longmapsto \mathbf{N} \setminus \{0\}$ be two functions. Suppose that $S \subseteq V(G)$. We say S is *a-satisfactory* if $d_S(x) \geq a(x)$ for any $x \in S$, and say S is *a-degenerate* if for any nonempty subset $S' \subseteq S$, there exists a vertex $x \in S'$ such that $d_{S'}(x) \leq a(x)$. Let (A, B) be a partition of G. Then (A, B) is an (a, b)-*degenerate partition* if A is a-degenerate and B is b-degenerate. A pair (X, Y) (not necessary a partition of G) of disjoint subsets X and Y of $V(G)$ is said to be (a, b)-*feasible pair* if X is a-satisfactory and Y is b-satisfactory. The following lemma, proved by Stiebitz [16], plays a key role in our proofs.

Lemma 2.1 (*Stiebitz [16]*)**.** *Let G be a graph and let $a, b : V(G) \longmapsto \mathbf{N}$ be two functions such that $d(x) \geq a(x) + b(x) - 1$ for each $x \in V(G)$. If G has an (a, b)-feasible pair, then it admits an (a, b)-feasible partition.*

The main idea of our proofs is an extension of that used in [16], where the *weight* of a partition is essential. For a partition (A, B) of a graph G, the weight $\omega(A, B)$ of (A, B) is defined by

$$\omega(A, B) = e(A) + e(B) + \sum_{u \in A} b(u) + \sum_{v \in B} a(v).$$

Then for $u \in A$, $v \in B$, a simple calculation shows that

$$\omega(A \setminus \{u\}, B \cup \{u\}) - \omega(A, B) = d_B(u) - d_A(u) + a(u) - b(u), \tag{1}$$

and

$$\omega(A \cup \{v\}, B \setminus \{v\}) - \omega(A, B) = d_A(v) - d_B(v) + b(v) - a(v). \tag{2}$$