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Saturation numbers for Ramsey-minimal graphs

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ABSTRACT

Given graphs $H_1, \ldots, H_t$, a graph G is $(H_1, \ldots, H_t)$-*Ramsey-minimal* if every t-coloring of the edges of G contains a monochromatic H_i in color i for some $i \in \{1, \ldots, t\}$, but any proper subgraph of G does not possess this property. We define $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ to be the family of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs. A graph G is $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-*saturated* if no element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of $G + e$. We define $sat(n, \mathcal{R}_{\min}(H_1, \ldots, H_t))$ to be the minimum number of edges over all $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs on n vertices. In 1987, Hanson and Toft conjectured that $sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = (r-2)(n-r+2)+\binom{r-2}{2}$ for $n \geq r$, where $r = r(K_{k_1}, \ldots, K_{k_t})$ is the classical Ramsey number for complete graphs. The first non-trivial case of Hanson and Toft's conjecture for sufficiently large n was settled in 2011, and is so far the only settled case. Motivated by Hanson and Toft's conjecture, we study the minimum number of edges over all $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$-saturated graphs on n vertices, where $\mathcal{T}_k$ is the family of all trees on k vertices. We show that for $n \geq 18$, $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) = \lfloor 5n/2 \rfloor$. For $k \geq 5$ and $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, we obtain an asymptotic bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ by showing that $\left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c \leq sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \leq \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n + C$, where $c = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right) k - 2$ and $C = 2k^2 - 6k + \frac{3}{2} - \left\lceil \frac{k}{2} \right\rceil \left(k - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - 1\right)$.

1. Introduction

All graphs considered in this paper are finite and without loops or multiple edges. For a graph G, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree, and $\overline{G}$ the complement of G. Given vertex sets $A, B \subseteq V(G)$, we say that A is *complete to* (resp. *anti-complete to*) B if for every $a \in A$ and every $b \in B$, $ab \in E(G)$ (resp. $ab \notin E(G)$). The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$, $e_G(A, B)$ the number of edges between A and B in G, and $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$, $e_G(a, B)$, and $G \setminus a$, respectively. For any edge $e \in E(\overline{G})$, we use $G + e$ to denote the graph obtained from G by adding the new edge e. The *join* $G \vee H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. For an integer $t \geq 1$ and a graph H, we define tH to be the union of t disjoint copies of H. We use K_n, $K_{1,n-1}$, C_n, P_n and T_n to denote the complete graph, star, cycle, path and a tree on n vertices, respectively.

Given graphs $G, H_1, \ldots, H_t$, we write $G \to (H_1, \ldots, H_t)$ if every t-edge-coloring of G contains a monochromatic H_i in color i for some $i \in \{1, 2, \ldots, t\}$. The classical *Ramsey number* $r(H_1, \ldots, H_t)$ is the minimum positive integer n such that

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$K_n \rightarrow (H_1, \ldots, H_t)$. A graph G is $(H_1, \ldots, H_t)$-*Ramsey-minimal* if $G \rightarrow (H_1, \ldots, H_t)$, but for any proper subgraph G' of G, $G' \not\rightarrow (H_1, \ldots, H_t)$. We define $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ to be the family of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs. It is straightforward to prove by induction that a graph G satisfies $G \rightarrow (H_1, \ldots, H_t)$ if and only if there exists a subgraph G' of G such that G' is $(H_1, \ldots, H_t)$-Ramsey-minimal. Ramsey's theorem [18] implies that $\mathcal{R}_{\min}(H_1, \ldots, H_t) \neq \emptyset$ for all integers t and all finite graphs $H_1, \ldots, H_t$. As pointed out in a recent paper of Fox, Grinshpun, Liebenau, Person, and Szabó [12], "it is still widely open to classify the graphs in $\mathcal{R}_{\min}(H_1, \ldots, H_t)$, or even to prove that these graphs have certain properties". Some properties of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ have been studied, such as the minimum degree $s(H_1, \ldots, H_t) := \min\{\delta(G) : G \in \mathcal{R}_{\min}(H_1, \ldots, H_t)\}$, which was first introduced by Burr, Erdős, and Lovász [4]. Recent results on $s(H_1, \ldots, H_t)$ can be found in [12,13]. For more information on Ramsey-related topics, the readers are referred to a very recent informative survey due to Conlon, Fox, and Sudakov [6].

In this paper, we study the following problem. A graph G is $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-*saturated* if no element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{R}_{\min}(H_1, \ldots, H_t)$ is a subgraph of $G+e$. This notion was initiated by Nešetřil [16] in 1986 when he asked whether there are infinitely many $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs. This was answered in the positive by Galluccio, Siminovits, and Simonyi [14]. We define $sat(n, \mathcal{R}_{\min}(H_1, \ldots, H_t))$ to be the minimum number of edges over all $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated graphs on n vertices. This notion was first discussed by Hanson and Toft [15] in 1987 when $H_1, \ldots, H_t$ are complete graphs. They proposed the following conjecture.

Conjecture 1.1. *Let $r = r(K_{k_1}, \ldots, K_{k_t})$ be the classical Ramsey number for complete graphs. Then*

$$sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ (r-2)(n-r+2) + \binom{r-2}{2} & n \geq r \end{cases}$$

Chen, Ferrara, Gould, Magnant, and Schmitt [5] proved that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10$ for $n \geq 56$. This settles the first non-trivial case of Conjecture 1.1 for sufficiently large n, and is so far the only settled case. Ferrara, Kim, and Yeager [11] proved that $sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_tK_2)) = 3(m_1 + \cdots + m_t - t)$ for $m_1, \ldots, m_t \geq 1$ and $n > 3(m_1 + \cdots + m_t - t)$. The problem of finding $sat(n, \mathcal{R}_{\min}(K_3, T_k))$ was also explored in [5].

Proposition 1.2. *Let $k \geq 2$ and $t \geq 2$ be integers. Then*

$$sat(n, \mathcal{R}_{\min}(K_t, T_k)) \leq n(t-2)(k-1) - (t-2)^2(k-1)^2 + \binom{(t-2)(k-1)}{2} + \left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r}{2},$$

where $r = n \pmod{k-1}$.

It was conjectured in [5] that the upper bound in Proposition 1.2 is asymptotically correct. Note that there is only one tree on three vertices, namely, P_3. A slightly better result was obtained for $\mathcal{R}_{\min}(K_3, P_3)$-saturated graphs in [5].

Theorem 1.3. *For $n \geq 11$, $sat(n, \mathcal{R}_{\min}(K_3, P_3)) = \left\lfloor \frac{5n}{2} \right\rfloor - 5$.*

Motivated by Conjecture 1.1, we study the following problem. Let $\mathcal{T}_k$ be the family of all trees on k vertices. Instead of fixing a tree on k vertices as in Proposition 1.2, we will investigate $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$, where a graph G is $(K_3, \mathcal{T}_k)$-*Ramsey-minimal* if for any 2-coloring $c : E(G) \rightarrow \{\text{red, blue}\}$, G has either a red K_3 or a blue tree $T_k \in \mathcal{T}_k$, and we define $\mathcal{R}_{\min}(K_3, \mathcal{T}_k)$ to be the family of $(K_3, \mathcal{T}_k)$-Ramsey-minimal graphs. By Theorem 1.3, we see that $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_3)) = \lfloor 5n/2 \rfloor - 5$ for $n \geq 11$. In this paper, we prove the following two main results. We first establish the exact bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4))$ for $n \geq 18$, and then obtain an asymptotic bound for $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ for all $k \geq 5$ and $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil + 2$.

Theorem 1.4. *For $n \geq 18$, $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_4)) = \left\lfloor \frac{5n}{2} \right\rfloor$.*

Theorem 1.5. *For any integers $k \geq 5$ and $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$, there exist constants $c = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right)k - 2$ and $C = 2k^2 - 6k + \frac{3}{2} - \left\lceil \frac{k}{2} \right\rceil \left(k - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - 1\right)$ such that*

$$\left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c \leq sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k)) \leq \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n + C.$$

The constants c and C in Theorem 1.5 are both quadratic in k. We believe that the true value of $sat(n, \mathcal{R}_{\min}(K_3, \mathcal{T}_k))$ is closer to the upper bound in Theorem 1.5. To establish the desired lower and upper bounds for each of Theorems 1.4 and 1.5, we need to introduce more notation and prove a useful lemma (see Lemma 1.6). Given a graph H, a graph G is H-*free* if G does not contain H as a subgraph. For a graph G, let $c : E(G) \rightarrow \{\text{red, blue}\}$ be a 2-edge-coloring of G and let E_r and E_b be the color classes of the coloring c. We use G_r and G_b to denote the spanning subgraphs of G with edge sets E_r and E_b, respectively.