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On trees with real-rooted independence polynomial

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ABSTRACT

The independence polynomial of a graph G is

$$I(G,x)=\sum_{k\geq 0} i_k(G)x^k,$$

where $i_k(G)$ denotes the number of independent sets of G of size k (note that $i_0(G) = 1$). In this paper we show a new method to prove real-rootedness of the independence polynomials of certain families of trees.

In particular we will give a new proof of the real-rootedness of the independence polynomials of centipedes (Zhu's theorem), caterpillars (Wang and Zhu's theorem), and we will prove a conjecture of Galvin and Hilyard about the real-rootedness of the independence polynomial of the so-called Fibonacci trees.

1. Introduction

The independence polynomial of a graph G is

$$I(G,x)=\sum_{k\geq 0} i_k(G)x^k,$$

where $i_k(G)$ denotes the number of independent sets of G of size k (note that $i_0(G) = 1$). In this paper we study the independence polynomials of trees. For trees, it is a well known conjecture that the sequence $(i_k(T))_{k\geq 0}$ is unimodal (see [1]).

Recall that a sequence $(b_k)_{k=0}^n$ is unimodal (see e.g. [7]), if there exists an index k, such that

$$b_0 \leq b_1 \leq \cdots \leq b_{k-1} \leq b_k \geq b_{k+1} \geq \cdots \geq b_n.$$

A stronger property for positive sequences is the so called log-concavity: for any i such that $0 < i < n$, we have $b_i^2 \geq b_{i-1}b_{i+1}$. An even stronger property is the real-rootedness of the polynomial $p(x) = \sum_{i=0}^n b_i x^i$ (any complex zero of the polynomial is real). This prompted many mathematicians to study trees with real-rooted independence polynomials. In this paper we show a general method to construct such trees or prove real-rootedness.

In particular we will give a new proof for real-rootedness of the independence polynomials of certain families of trees, which includes centipedes (Zhu's theorem, see [13]), caterpillars (Wang and Zhu's theorem, see [9]), and we will prove a conjecture of Galvin and Hilyard about the real-rootedness of the independence polynomial of the Fibonacci trees (see Section 6 of [4]).

Recall that the n-centipede W_n is a graph (Fig. 1a), such that we take a path on n vertices and we hang 1 pendant edge from each of its vertices. Similarly the n-caterpillar H_n is the graph (Fig. 1b) obtained by taking a path on n vertices and by

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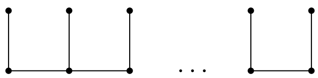
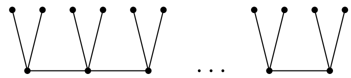
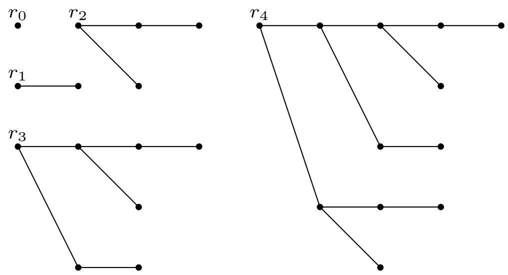


(a) The centipede

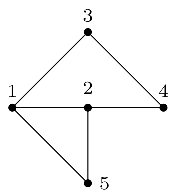


(b) The caterpillar

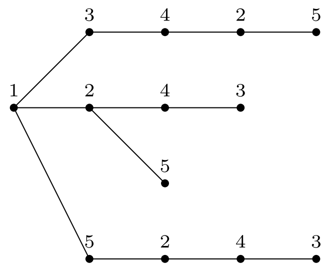


(c) The first 5 Fibonacci trees

Fig. 1. Some families of trees.



(a) A graph G with labelled vertices.



(b) Path tree of G from vertex 1. The labels of the vertices denote endpoints of paths.

Fig. 2. A graph with its path tree.

hanging 2 pendant edges from each of its vertices. The Fibonacci trees were defined by Wagner in [8] as follows (Fig. 1c): let $F_0 = K_1$ and $F_1 = K_2$ with roots $r_0 \in V(F_0)$ and $r_1 \in V(F_1)$. Then for $n \geq 2$ the nth Fibonacci tree F_n is obtained by the disjoint union of F_{n-1}, F_{n-2} and a new vertex, labelled by r_n, and by connecting r_n to the roots of F_{n-1} and F_{n-2}. Define r_n as the root of F_n.

1.1. Methods and motivations

To motivate our method we will use certain results from the theory of matching polynomials. Recall that the matching polynomial of a graph G is defined as:

$$\mu(G, x) = \sum_{k \geq 0} (-1)^k m_k(G) x^{n-2k},$$

where $m_k(G)$ is the number of matchings with k edges (note that $m_0(G) = 1$). One of the best known theorems about matching polynomials is that for any finite graph G and $u \in V(G)$ there exists a rooted tree (T, r), such that

$$\frac{\mu(G-u, x)}{\mu(G, x)} = \frac{\mu(T-r, x)}{\mu(T, x)}. \tag{1}$$

A well-known construction for T is the path-tree [5] (a.k.a. Godsil tree), which is the tree on paths of G starting from u, and the edges are the strict inclusions. (For an example see Fig. 2.)

In this paper we will prove an "independence version" of this theorem through a quite similar construction. More precisely, we will show that there exists a rooted tree (T', r), such that

$$\frac{I(G-u, x)}{I(G, x)} = \frac{I(T'-r, x)}{I(T', x)}.$$