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Self-dual codes from orbit matrices and quotient matrices of combinatorial designs

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ABSTRACT

In this paper we give constructions of self-orthogonal and self-dual codes, with respect to certain scalar products, with the help of orbit matrices of block designs and quotient matrices of symmetric (group) divisible designs (SGDDs) with the dual property. First we describe constructions from block designs and their extended orbit matrices, where the orbit matrices are induced by the action of an automorphism group of the design. Further, we give some further constructions of self-dual codes from symmetric block designs and their orbit matrices. Moreover, in a similar way as for symmetric designs, we give constructions of self-dual codes from SGDDs with the dual property and their quotient matrices.

1. Introduction

Throughout the paper the terminology and notation regarding block designs and codes follow [1,4,20]. For more information about divisible designs we refer the reader to [6].

Let $C \subseteq \mathbb{F}_p^n$ be a linear code. Its *dual code* is the code $C^\perp = \{x \in \mathbb{F}_p^n | x \cdot c = 0, \ \forall c \in C\}$, where $\cdot$ is the standard inner product. The code C is called *self-orthogonal* if $C \subseteq C^\perp$, and C is called *self-dual* if $C = C^\perp$. We may use a symmetric nonsingular matrix U over the field $\mathbb{F}_p$ to define a scalar product $\langle \cdot, \cdot \rangle_U$ for row vectors in $\mathbb{F}_p^n$: $\langle a, c \rangle_U = aUc^T$ (see [14,21]). The *U-dual code* of a linear code C is the code $C^U = \{a \in \mathbb{F}_p^n \mid \langle a, c \rangle_U = 0, \ \forall c \in C\}$. A code C is called *self-U-dual*, or self-dual with respect to U, if $C = C^U$.

Given an $m \times n$ integer matrix A, we denote by $row_{\mathbb{F}}(A)$ the linear code over the field $\mathbb{F}$ spanned by the rows of A, and by $col_{\mathbb{F}}(A)$ we denote the linear code over the field $\mathbb{F}$ spanned by the columns of A. By $row_p(A)$ and $col_p(A)$ we denote the p-ary linear codes spanned by the rows of A and columns of A, respectively.

In this paper we construct codes from orbit matrices of block designs and quotient matrices of symmetric group divisible designs (SGDDs) with the dual property, which are self-dual or self-orthogonal with respect to certain scalar products.

The paper is organized as follows. In Section 2, we will define orbit matrices of block designs and quotient matrices of symmetric group divisible designs (SGDDs) with the dual property. In Section 3, we will introduce several constructions of self-orthogonal and self-dual codes from extended orbit matrices of block designs (that are not necessarily symmetric). A starting point, which gave ideas for these constructions, was Theorem 25 from [21] (given here as Theorem 3.2) that describes a construction of self-dual codes from incidence matrices of block designs. In Section 4, we give some more constructions of self-dual codes related particularly to orbit matrices of symmetric block designs and also to quotient matrices of SGDDs with the dual property. This paper can also be regarded as a generalization of the results presented in [8] and [10].

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2. Orbit matrices and quotient matrices

In this section, we give definitions and basic facts on orbit matrices of block designs and quotient matrices of symmetric group divisible designs (SGDDs) with the dual property, that will be used in the rest of the paper.

2.1. Orbit matrices of block designs

A 2-(v, k, λ) *design* is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, such that $|\mathcal{P}| = v$, any element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$, and any two elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$. The elements of the set $\mathcal{P}$ are called *points*, and the elements of the set $\mathcal{B}$ are called *blocks*. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-(v, k, λ) design and let $G \leq Aut(\mathcal{D})$. We will denote with $P_1, \dots, P_n$ G-orbits of points, with $B_1, \dots, B_m$ G-orbits of blocks and let $|P_i| = \omega_i$, $|B_j| = \Omega_j$, where $1 \leq i \leq n$, $1 \leq j \leq m$. For the block $x \in \mathcal{B}$ and the point $Q \in \mathcal{P}$ we will introduce the following notation: $\langle x \rangle = \{R \in \mathcal{P} | (R, x) \in I\}$, $\langle Q \rangle = \{y \in \mathcal{B} | (Q, y) \in I\}$.

Let $Q \in P_i$, $x \in B_j$. We will denote:

$$\gamma_{ij} = |\langle x \rangle \cap P_i|, \quad \Gamma_{ij} = |\langle Q \rangle \cap B_j|.$$

It is known that G-orbits of points and blocks give a tactical decomposition of the design $\mathcal{D}$ (see [3]). Therefore the numbers γ_{ij} do not depend on the choice of the block x, just like the numbers Γ_{ij} do not depend on the choice of the point Q as the representative of the point orbit P_i. The number γ_{ij} equals the number of points from the point orbit P_i that are incident with the block from the block orbit B_j. Similarly, Γ_{ij} equals the number of blocks from the block orbit B_j that are incident with the point from the point orbit P_i. Therefore:

$$\sum_{i=1}^{n} \gamma_{ij} = k, \ \forall j \in \{1, \dots, m\}, \qquad \sum_{j=1}^{m} \Gamma_{ij} = r, \ \forall i \in \{1, \dots, n\}.$$

The proof of the following lemma can be found in [9].

Lemma 2.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a block design, $G \leq Aut(\mathcal{D})$, and let ω_i, Ω_j, γ_{ij}, Γ_{ij} be defined as before. The following equations hold:*

(a) $\Omega_j \gamma_{ij} = \omega_i \Gamma_{ij}$;

(b) $\sum_{j=1}^{m} \Gamma_{ij} \gamma_{sj} = \lambda \omega_s + \delta_{is} \cdot (r - \lambda)$, *where δ_{is} is the Kronecker delta, $i, s \in \{1, \dots, n\}$.*

The following proposition can be obtained using Lemma 2.1.

Proposition 2.1 ([9]). *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a block design, $G \leq Aut(\mathcal{D})$, and let ω_i, Ω_j, γ_{ij}, Γ_{ij} be defined as before. The following equations hold:*

1. $\sum_{i=1}^{n} \gamma_{ij} = k$;
2. $\sum_{j=1}^{m} \frac{\Omega_j}{\omega_i} \gamma_{ij} \gamma_{sj} = \lambda \omega_s + \delta_{is} \cdot (r - \lambda)$.

We will now define point and block orbit matrices as they are defined in [7].

Definition 2.1. A *point orbit matrix* for parameters (v, k, λ) and the orbit length distribution for points $\omega = (\omega_1, \dots, \omega_n)$, and for blocks $\Omega = (\Omega_1, \dots, \Omega_m)$ is any $n \times m$ matrix $S = [\Gamma_{ij}]$ with the elements from $\mathbb{N}_0$ satisfying the following properties:

1. $0 \leq \Gamma_{ij} \leq \Omega_j$, $1 \leq i \leq n$, $1 \leq j \leq m$; 2. $\sum_{j=1}^{m} \Gamma_{ij} = r$, $1 \leq i \leq n$;

3. $\sum_{i=1}^{n} \frac{\omega_i}{\Omega_j} \Gamma_{ij} = k$, $1 \leq j \leq m$; 4. $\sum_{j=1}^{m} \frac{\omega_t}{\Omega_j} \Gamma_{sj} \Gamma_{tj} = \begin{cases} \lambda \omega_t, & s \neq t, \\ \lambda(\omega_t - 1) + r, & s = t. \end{cases}$

Then it holds that: $\sum_{i=1}^{n} \omega_i = v$, $\sum_{j=1}^{m} \Omega_j = b$, $r = \frac{v-1}{k-1}\lambda$ and $b = \frac{vr}{k}$.