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Note

A note on the list vertex arboricity of toroidal graphs

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ABSTRACT

The vertex arboricity $a(G)$ of a graph G is the minimum number of colors required to color the vertices of G such that no cycle is monochromatic. The list vertex arboricity $a_l(G)$ is the list-coloring version of this concept. In this note, we prove that if G is a toroidal graph, then $a_l(G) \leq 4$; and $a_l(G) = 4$ if and only if G contains K_7 as an induced subgraph.

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1. Introduction

All graphs considered in this note are finite, simple and undirected. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$ to denote its vertex set, edge set, maximum degree, and minimum degree, respectively. For a vertex $v \in V(G)$, let $d_G(v)$ denote the degree of v in G and $N_G(v)$ the set of neighbors of v in G. A graph G is called *planar* (or *toroidal*) if it can be embedded in the plane (or in the torus) such that any two edges intersect only at their ends. Here we assume that all embeddings on the surface are 2-cell embeddings. In other words, each face of G is homeomorphic to an open disk. If G has an embedding on the surface, then we denote by $F(G)$ the set of faces with respect to this embedding.

The *vertex arboricity*, denoted by $a(G)$, of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a forest. Obviously, $a(G) = 1$ if and only if G itself is a forest.

In 1968, Chartrand, Kronk and Wall [2] first introduced the vertex arboricity of a graph and proved that $a(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph G and $a(G) \leq 3$ for any planar graph G. It is known that there exist infinitely many planar graphs G such that $a(G) = 3$. In particular, Hakimi and Schmeichel [7] made a characterization that a plane graph G has $a(G) = 2$ if and only if G^*, the dual of G, contains a connected Eulerian spanning subgraph. Some sufficient conditions for a planar graph G to have $a(G) \leq 2$ have recently been given in [4,9,12,14].

List-coloring, in which each element is colored by a color from its own list of colors, was introduced independently by Vizing [13] in 1976 and by Erdős et al. [6] in 1980. In a very natural way, Borodin et al. [1] combined the concepts of vertex arboricity and list-coloring to introduce the *list vertex arboricity* of graphs. A graph G is called *list vertex k-arborable* if for any list $L(v)$ of cardinality at least k at each vertex v of G, one can choose a color for each v from its list $L(v)$ so that the subgraph induced by every color class is a forest. The smallest k for a graph to be list vertex k-arborable is denoted by $a_l(G)$. Obviously, $a(G) \leq a_l(G)$ for any graph G.

Suppose that G is a toroidal graph. Then it was shown in [10] that $a(G) \leq 4$, and in [11] that $a(G) \leq 2$ if moreover G does not contain 3-cycle. Recently, Choi and Zhang [5] obtained a parallel result that $a(G) \leq 2$ if G does not contain 4-cycle. For the list vertex arboricity of toroidal graphs, the following several results have been affirmed:

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• $a_l(G) \leq 2$ if G contains no 5-cycle [15].
• $a_l(G) \leq 2$ if G contains no 3-cycle adjacent to a 4-cycle [3].
• $a_l(G) \leq 2$ if G contains no 3-cycle adjacent to a 5-cycle [8].

In this note, we will show that every toroidal graph G satisfies $a_l(G) \leq 4$, and the equality holds if and only if G contains K_7, the complete graph of order 7, as an induced subgraph.

2. Results

To obtain our main result, we first investigate the structural properties of toroidal graphs.

Lemma 1. *Let G be a connected toroidal graph with $\delta(G) = 6$. Then the following statements hold:*
(1) *G is 6-regular.*
(2) $|E(G)| = 3|V(G)|$.
(3) *G is triangulated. Namely, each face of G is of degree 3.*

Proof. Since G is a toroidal graph, it is well known that $|E(G)| \leq 3|V(G)|$. In each of following cases, we will make use of contradictions to prove the statements (1)–(3).

(1) Assume to the contrary that there exists a vertex $v^* \in V(G)$ with $d_G(v^*) \geq 7$. By applying the handshaking lemma

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)|,$$

one may easily derive that

$$2|E(G)| = \sum_{v \in V(G)} d_G(v) = d_G(v^*) + \sum_{v \in V(G) \setminus \{v^*\}} d_G(v) \geq 7 + 6(|V(G)| - 1) = 6|V(G)| + 1.$$

Hence

$$|E(G)| \geq 3|V(G)| + \frac{1}{2},$$

which is a contradiction.

(2) Assume to the contrary that $|E(G)| < 3|V(G)|$. By Lemma 1(1), we deduce that

$$\sum_{v \in V(G)} d_G(v) = 6|V(G)| > 2|E(G)|,$$

which is impossible.

(3) Assume to the contrary that there exists a face $f^* \in F(G)$ with $d_G(f^*) \geq 4$. Using the Euler's formula

$$|V(G)| - |E(G)| + |F(G)| = 0,$$

we may obtain a series of inequalities:

$$2|E(G)| = \sum_{f \in F(G)} d_G(f) \geq 4 + 3(|F(G)| - 1) = 3|F(G)| + 1 = 3(|E(G)| - |V(G)|) + 1.$$

This implies immediately that

$$|E(G)| \leq 3|V(G)| - 1,$$

which contradicts Lemma 1(2). □

The following notation, which will apply to the proof of the main result, deals with a *partial list vertex k-arborable coloring* of a graph G, defined as a mapping ϕ from list assignment L to a subset V' of $V(G)$. Given a partial list vertex L-arborable coloring ϕ with v being uncolored, we say that v can be *safely colored* on the basic of ϕ if there is a color $c \in L(v)$ which appears at most once among all its colored neighbors. Such color c is also said to be *safe* for v.

For an integer $k \geq 1$, a graph G is called *k-degenerate* if every subgraph H of G contains a vertex u such that $d_H(u) \leq k$. The following consequence was provided by Zhang in [15].

Lemma 2. *If G is a k-degenerate graph with $k \geq 1$, then $a_l(G) \leq \lceil \frac{k+1}{2} \rceil$.*

Before establishing our main theorem, we need one more lemma whose proof is not difficult. However, we still would like to present a proof for it here, for the sake of completeness.

Lemma 3. *Let K_n be a complete graph of order $n \geq 2$. Then $a_l(K_n) = \lceil \frac{n}{2} \rceil$.*

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