Discrete Mathematics 341 (2018) 3355–3369

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Shedding vertices of vertex decomposable well-covered graphs

Jonathan Baker [a], Kevin N. Vander Meulen [b], Adam Van Tuyl [c,*]

[a] *Brock University, Hamilton, ON, L8K 1V7, Canada*
[b] *Department of Mathematics Redeemer University College, Ancaster, ON, L9K 1J4, Canada*
[c] *Department of Mathematics and Statistics, McMaster University, Hamilton, ON, L8S 4L8, Canada*

ARTICLE INFO

Article history:
Received 20 February 2017
Received in revised form 24 July 2018
Accepted 27 July 2018

Keywords:
Well-covered graph
Vertex decomposable graph
Dominating set
Shedding vertices
Cohen–Macaulay graph

ABSTRACT

We focus our attention on well-covered graphs that are vertex decomposable. We show that for many known families of these vertex decomposable graphs, the set of shedding vertices forms a dominating set. We then construct three new infinite families of well-covered graphs, none of which have this property. We use these results to provide a minimal counterexample to a conjecture of Villarreal regarding Cohen–Macaulay graphs.
© 2018 Elsevier B.V. All rights reserved.

1. Introduction

In this paper we focus on well-covered graphs G that have the additional property of being vertex decomposable (see Definition 2.1). A subset D of the vertex set V of G is a *dominating set* if every vertex $x \in V \setminus D$ is adjacent to a vertex of D. We observe that for most of the known constructions of pure vertex decomposable graphs, the set of shedding vertices $\mathrm{Shed}(G)$ is a dominating set. The next result summarizes some of our findings.

Theorem 1.1. *Suppose that G is a pure vertex decomposable graph. If G is*

(i) *a bipartite graph, or*
(ii) *a chordal graph, or*
(iii) *a very well-covered graph, or*
(iv) *a vertex-transitive graph, or*
(v) *a Cameron–Walker graph, or*
(vi) *a clique-whiskered graph, or*
(vii) *a graph with girth at least five,*

then $\mathrm{Shed}(G)$ *is a dominating set.*

In particular, (i) is Corollary 6.4, (ii) is Theorem 4.3, (iii) is Theorem 6.3, (iv) is Theorem 4.1, (v) is Corollary 5.2, (vi) is Theorem 5.3, and (vii) Theorem 7.3.

* Corresponding author.
E-mail addresses: jb16ok@brocku.ca (J. Baker), kvanderm@redeemer.ca (K.N. Vander Meulen), vantuyl@math.mcmaster.ca (A. Van Tuyl).

https://doi.org/10.1016/j.disc.2018.07.029
0012-365X/© 2018 Elsevier B.V. All rights reserved.

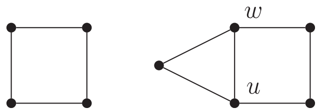

Fig. 1. Two well-covered graphs.

The fact that Shed(G) is a dominating set for all these known vertex decomposable graphs led us to question if this is a feature of all pure vertex decomposable graphs. Pursuing that question eventually led us to develop three new infinite families of (vertex decomposable) well-covered graphs. These infinite families fail to have the property that Shed(G) is a dominating set and, as we show at the end of the paper, provide new counterexamples and insight to a conjecture of Villarreal.

We outline the structure of this paper. Section 2 introduces the definition of pure vertex decomposable graphs and Section 3 describes the set of shedding vertices with some introductory tools for identifying them. Section 4 develops our results for the chordal and vertex-transitive pure vertex decomposable graphs. In Section 5, we consider two constructions of pure vertex decomposable graphs, and show that any pure vertex decomposable graph G constructed via either construction satisfies the property that Shed(G) is a dominating set. In Section 6, we consider all the very well-covered graphs that are vertex decomposable. In Section 7, we focus on all pure vertex decomposable graphs with girth at least five. In Section 8, we describe three infinite families of graphs where each graph G is pure vertex decomposable, but Shed(G) is not a dominating set. In Section 9, we show how to take a graph G which is pure vertex decomposable but Shed(G) is not a dominating set and duplicate a vertex to construct a larger graph with the same properties. We conclude with Section 10, describing how our results provide new counterexamples for a conjecture of Villarreal. Via a computer search, we find the smallest pure vertex decomposable graph G for which Shed(G) is not a dominating set. As part of our computer search, we also show that the set of pure vertex decomposable graphs is the same as the set of Cohen–Macaulay graphs for all the graphs on 10 vertices or fewer. The facts that a minimal counterexample requires at least nine vertices and that the standard constructions, as described in Theorem 1.1, do not provide any counterexamples, make the new constructions in Section 8 relevant for any further analysis of the relationship between dominating sets and vertex decomposability.

2. Vertex decomposable graphs

Let G be a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E. We may sometimes write $V(G)$, respectively $E(G)$, for V, respectively E, if we wish to highlight that we are discussing the vertices, respectively edges, of G. A subset $W \subseteq V$ is an *independent* set if no two vertices of W are adjacent. An independent set W is a *maximal independent set* if there is no independent set U such that W is a proper subset of U. If $W \subseteq V$ is an independent set, then $V \setminus W$ is a *a vertex cover*. A vertex cover C is a *minimal vertex cover* if $V \setminus C$ is a maximal independent set. A graph is *well-covered* if all the maximal independent sets have the same cardinality, or equivalently, if every minimal vertex cover has the same cardinality. For example, if P_n is the path graph on $n \geq 2$ vertices, then P_n is well-covered if and only if $n = 2$ or $n = 4$. The graphs in Fig. 1 are well-covered graphs.

For any $x \in V$, let $G \setminus x$ denote the graph G with the vertex x and incident edges removed. The collection of *neighbours* of a vertex $x \in V$ in G, is the set $N(x) = \{y \mid \{x, y\} \in E\}$. The *closed neighbourhood* of a vertex x is $N[x] = N(x) \cup \{x\}$. We sometimes write $N_G(x)$ or $N_G[x]$ to highlight which graph G we are considering. For $S \subseteq V$, we let $G \setminus S$ denote the graph obtained by removing all the vertices of S and their incident edges.

Definition 2.1. A graph G is *pure vertex decomposable* if G is well-covered and

(i) G consists of isolated vertices, or G is empty, or
(ii) there exists a vertex $x \in V$, called a *shedding vertex*, such that $G \setminus x$ and $G \setminus N[x]$ are pure vertex decomposable.

For example, the first graph, C_4, in Fig. 1 is not pure vertex decomposable since the deletion on any vertex gives the path P_3 which is not well-covered and hence not pure vertex decomposable. The second graph G in Fig. 1 is pure vertex decomposable: $G \setminus u$ is the pure vertex decomposable graph P_4 and $G \setminus N[u]$ is an isolated vertex.

If G is pure vertex decomposable, then the set of shedding vertices is denoted by:

$$\text{Shed}(G) = \{x \in V \mid G \setminus x \text{ and } G \setminus N[x] \text{ are pure vertex decomposable}\}.$$

For example, Shed(G) $= \{u, w\}$ for the pure vertex decomposable graph in Fig. 1.

Remark 2.2. The study of vertex decomposable graphs lies in the intersection of combinatorial algebraic topology and combinatorial commutative algebra. In particular, Dochtermann–Engström [8] and Woodroofe [26] independently showed that vertex decomposability of an independence complex is a useful tool for exploring algebraic properties of an edge ideal of a graph. The *independence complex* of a graph G, denoted Ind(G), is the simplicial complex

$$\text{Ind}(G) = \{W \subseteq V \mid W \text{ is an independent set}\}.$$

Download English Version:

https://daneshyari.com/en/article/9514439

Download Persian Version:

https://daneshyari.com/article/9514439

Daneshyari.com