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Congruences modulo powers of 3 for 3- and 9-colored generalized Frobenius partitions

Liuquan Wang

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei, People's Republic of China

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ABSTRACT

Let $c\phi_k(n)$ be the number of k-colored generalized Frobenius partitions of n. We establish some infinite families of congruences for $c\phi_3(n)$ and $c\phi_9(n)$ modulo arbitrary powers of 3, which refine the results of Kolitsch. For example, for $k \ge 3$ and $n \ge 0$, we prove that

$$c\phi_3\left(3^{2k}n + \frac{7 \cdot 3^{2k}+1}{8}\right) \equiv 0 \pmod{3^{4k+5}}.$$

We give two different proofs to the congruences satisfied by $c\phi_9(n)$. One of the proofs uses a relation between $c\phi_9(n)$ and $c\phi_3(n)$ due to Kolitsch, for which we provide a new proof in this paper.

1. Introduction

In 1984, Andrews [1] introduced the concept of k-colored generalized Frobenius partitions. We first color all the nonnegative integers using "colors" denoted by $1, 2, \dots, k$, and we write j_i as the integer j colored with the ith color. We say that $j_i < h_s$ if and only if $j < h$ or $j = h$ and $i < s$. Now we have an ordering on these colored integers as follows:

$$0_1 < 0_2 < \cdots < 0_k < 1_1 < 1_2 < \cdots < 1_k < 2_1 < 2_2 < \cdots < 2_k < \cdots. \tag{1.1}$$

A k-colored generalized Frobenius partition is a two-row array of colored integers of the form

$$\begin{pmatrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{pmatrix}$$

such that

$$a_r < a_{r-1} < \cdots < a_1, \quad b_r < b_{r-1} < \cdots < b_1. \tag{1.2}$$

The number being partitioned by this partition is

$$n = r + \sum_{i=1}^{r}(a_i + b_i).$$

For any positive integer k, Andrews [1] used the symbol $c\phi_k(n)$ to denote the number of k-colored generalized Frobenius partitions of n. Let

$$\mathrm{C}\Phi_k(q) := \sum_{n=0}^{\infty} c\phi_k(n)q^n.$$

E-mail address: wanglq@whu.edu.cn.

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Andrews [1] proved that

$$\mathrm{C}\Phi_k(q) = \frac{1}{(q;q)_\infty^k} \sum_{m_1,\ldots,m_{k-1}=-\infty}^{\infty} q^{Q(m_1,\ldots,m_{k-1})}, \tag{1.3}$$

where

$$Q(m_1,\ldots,m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1\le i<j\le k-1} m_i m_j$$

and

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1.$$

To investigate arithmetic properties of $c\phi_k(n)$, Andrews [1] obtained alternative representations for $\mathrm{C}\Phi_k(q)$ with $k \in \{2,3,5\}$. In particular, for $k=3$, he [1, Eq. (9.4)] proved that

$$\sum_{n=0}^{\infty} c\phi_3(n)q^n = \frac{1}{(q;q)_\infty^3}\left(1+6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\right). \tag{1.4}$$

From the formulas of $\mathrm{C}\Phi_k(q)$, Andrews [1] found some nice properties of $c\phi_k(n)$. For instance, he proved that for $n \ge 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod{5}. \tag{1.5}$$

Since then, many congruences satisfied by $c\phi_k(n)$ have been discovered. Sellers [21] conjectured that (1.5) can be extended to a congruence modulo arbitrary powers of 5. Namely, for any integers $k \ge 1$ and $n \ge 0$, he conjectured that

$$c\phi_2\left(5^k n + \lambda_k\right) \equiv 0 \pmod{5^k}, \tag{1.6}$$

where λ_k is the reciprocal of 12 modulo 5^k. This conjecture was later proved by Paule and Radu [20] using the theory of modular forms.

After the work of Andrews, Kolitsch [13,14] introduced the function $\overline{c\phi}_k(n)$, which denotes the number of k-colored generalized Frobenius partitions of n whose order is k under cyclic permutation of the colors. He [14] proved that for any positive integer m,

$$\overline{c\phi}_m(n) = \sum_{d|(m,n)} \mu(d) c\phi_{\frac{m}{d}}\left(\frac{n}{d}\right), \tag{1.7}$$

where $\mu(x)$ is the Möbius function. In particular, when m is a prime, we have

$$\overline{c\phi}_m(n) = c\phi_m(n) - p\left(\frac{n}{m}\right), \tag{1.8}$$

where $p(n)$ is the ordinary partition function and we agree that $p(x)=0$ when x is not an integer. Let t_k be the reciprocal of 8 modulo 3^k. Kolitsch [15] established the following infinite families of congruences: for $k \ge 1$ and $n \ge 0$,

$$\overline{c\phi}_3(3^k n + t_k) \equiv 0 \begin{cases} \pmod{3^{2k+2}} & \text{if } k \text{ is even,} \\ \pmod{3^{2k+1}} & \text{if } k \text{ is odd.} \end{cases} \tag{1.9}$$

From (1.8) we see that $c\phi_3(n) = \overline{c\phi}_3(n)$ if n is not divisible by 3. Thus (1.9) implies that for $k \ge 1$ and $n \ge 0$,

$$c\phi_3\left(3^{2k-1}n + \frac{5\cdot 3^{2k-1}+1}{8}\right) \equiv 0 \pmod{3^{4k-1}}, \tag{1.10}$$

$$c\phi_3\left(3^{2k}n + \frac{7\cdot 3^{2k}+1}{8}\right) \equiv 0 \pmod{3^{4k+2}}. \tag{1.11}$$

In 1996, by using some combinatorial arguments, Kolitsch [16, Theorem 2] proved that for any $n \ge 1$,

$$\overline{c\phi}_9(n) = 3\overline{c\phi}_3(3n-1). \tag{1.12}$$

Using (1.7), this relation is equivalent to

$$c\phi_9(n) = 3c\phi_3(3n-1) + c\phi_3\left(\frac{n}{3}\right). \tag{1.13}$$