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The bi-join decomposition

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Abstract

We introduce a new generalization of the modular decomposition called the *bi-join decomposition*. We characterize the completely decomposable graphs and we give a linear-time decomposition algorithm.

Keywords: decomposition, modular decomposition, graph classes.

1 Introduction

Graph decompositions are widely used in graph algorithms and graph theory. Well-known examples are the modular decomposition, and its generalization the split decomposition. Both correspond to a decomposition tree, whose leaves are the graph vertices and whose internal nodes correspond to decomposition operations. When the degree of the tree is bounded, some NP-hard problems can be solved in linear-time using recursivity on the tree [8]. Modular decomposition has been studied for a long time. Some properties of the modular decomposition extend to the split decomposition, including unicity [4] and linear-time computation of the decomposition tree [5].

For a decomposition $\mathcal{D}$ we say that a graph G is *completely decomposable* if every subgraph large enough (with at least four or five vertices, depending on the decomposition) admits a non-trivial decomposition, and on the other side G is *prime* if it admits no non-trivial decomposition (the meaning of "trivial" of course depends on the decomposition).

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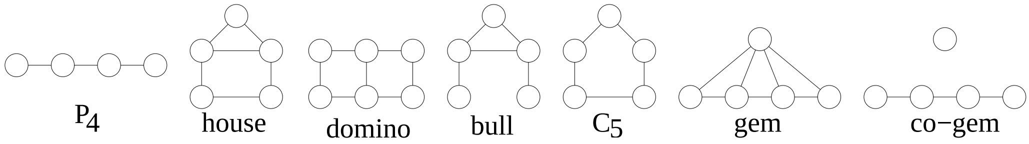

A *module* M of a graph $G = (V, E)$ is a set of vertices such that a vertex $v \notin M$ is adjacent either to all vertices of M or to none of them. The modular decomposition is the set of all modules of a graph [8]. We do not discuss this notion here but just recall the following famous theorem from [2].

Theorem 1.1 *The following propositions are equivalent:*
(1) G is completely decomposable w.r.t. modular decomposition.
(2) G is P_4-free.
(3) For every subgraph H of G, either H or $\overline{H}$ are not connected.
(4) G can be obtained from a single vertex by a sequence of extensions by a true twin or a false twin.

These graphs are called *cographs*. *Splits* are a generalization of modules. A *distance hereditary graph* is a graph in which all chordless paths between any two vertices have the same length. There is also a nice characterization theorem from [1,6].

Theorem 1.2 *The following propositions are equivalent:*
(1) G is completely decomposable w.r.t. split decomposition.
(2) G is a distance hereditary graph.
(3) G is (house,hole,domino,gem)-free.
(4) G can be obtained from a single vertex by a sequence of extensions by a true twin or by a false twin or a pendant vertex.

In this paper, we introduce a new generalization of the modular decomposition called the *bi-join decomposition*. We show that some properties of the modular and the split decomposition are preserved, we characterize the completely decomposable graphs and we give a linear-time decomposition algorithm.

2 Bi-joins and the bi-join decomposition

A *bipartition* of a set V is an (unordered) pair $\{V_1, V_2\}$ such that $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. Two bipartitions of V $\{V_1, V_2\}$ and $\{V_3, V_4\}$ *overlap* if the four sets $V_1 \cap V_3$, $V_2 \cap V_3$, $V_1 \cap V_4$ and $V_2 \cap V_4$ are nonempty. In a graph, there is a *complete join* between $A \subseteq V$ and $B \subseteq V$ if every vertex of A is linked with every vertex of B.