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Electronic Notes in Discrete Mathematics 22 (2005) 233–237
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

Paired-domination of Cartesian products of graphs and rainbow domination

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Abstract

The most famous open problem involving domination in graphs is Vizing's conjecture which states the domination number of the Cartesian product of any two graphs is at least as large as the product of their domination numbers. We investigate a similar problem for paired-domination, and obtain a lower bound in terms of product of domination number of one factor and 3-packing of the other factor. Some results are obtained by applying a new graph invariant called rainbow domination.

Keywords: graph products, paired-domination, 3-packing, Vizing's conjecture.

doi:10.1016/j.endm.2005.06.059

1 Paired-domination of Cartesian product of graphs

A paired-dominating set of a graph G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph induced by S contains a perfect matching. The paired-domination number $\gamma_{\rm pr}(G)$ of G is the minimum cardinality of a paired-dominating set.

For graphs G and H, the *Cartesian product* $G \,\Box\, H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. In 1968 Vizing [8] made the following conjecture which he first posed as a question in 1963.

Conjecture 1.1 (Vizing's Conjecture) *For any graphs G and H, $\gamma(G)\gamma(H) \leq \gamma(G \,\Box\, H)$.*

Vizing's Conjecture has yet to be settled (although the conjecture has been proven true for large classes of graphs [2]). The best general upper bound to date on the product of the domination numbers of two graphs in terms of their Cartesian product is due to Clark and Suen [1].

Theorem 1.2 (Clark, Suen [1]) *For any graphs G and H, $\gamma(G)\gamma(H) \leq 2\gamma(G \,\Box\, H)$.*

Haynes and Slater [4] proved that for any graph G without isolated vertices, $\gamma_{\rm pr}(G) \leq 2\gamma(G)$. Combining this with Theorem 1.2 we obtain for graphs G and H without isolated vertices, $\gamma_{\rm pr}(G)\gamma_{\rm pr}(H) \leq 4\gamma(G)\gamma(H) \leq 8\gamma(G \,\Box\, H) \leq 8\gamma_{\rm pr}(G \,\Box\, H)$. This general bound can be improved if one of the graphs has a special structure with respect to packings in graphs.

For $k \geq 2$, Meir and Moon [7] defined a *k-packing* in a graph G as a set S of vertices of G that are pairwise at distance greater than k apart, i.e., if $u, v \in S$, then $d_G(u, v) > k$. The *k-packing number* of G, denote $\rho_k(G)$, is the maximum cardinality of a k-packing in G. A graph G is called a (ρ_2, γ)-graph if $\rho_2(G) = \gamma(G)$.

We are aiming (see Theorem 1.7) at a similar bound as obtained by Henning and Rall [5] for the total domination. For any graphs G and H without isolated vertices, at least one of which is a (ρ_2, γ)-graph,

$$\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \,\Box\, H).$$

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