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Electronic Notes in Discrete Mathematics 22 (2005) 507–513
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

Optimal Partition of a Bipartite Graph with Prescribed Layout into Non-Crossing b-Matchings

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Abstract

In this paper we deal with the problem of partitioning the edge set of a bipartite graph $G = (L \cup R, E)$ with prescribed layout into the minimum number of non-crossing b-matchings. Some bounds and properties are discussed and an exact $O(|E| \log \log \min\{|L|, |R|\})$ is presented for its solution.

Keywords: Matching, colouring, bipartite graphs.

1 Introduction

In this paper we deal with the problem $\mathcal{P}$ of partitioning the edge set of a bipartite graph with prescribed layout into the minimum number of non-

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doi:10.1016/j.endm.2005.06.091

crossing b-matchings. $\mathcal{P}$ depends on both the topology of a bipartite graph, on the geometry defined by a given layout of it, and on the array b.

The topology of the bipartite graph is fully represented by $G = (L \cup R, E)$, where $L = \{l_1, l_2, ..., l_{|L|}\}$ and $R = \{r_1, r_2, ..., r_{|R|}\}$ are two independent sets of nodes, and E is the set of edges, each of which is an ordered pair (l_i, r_j) of nodes. As for the geometry of the problem, we shall assume that a layout λ is given, and precisely: the nodes in L, the *left nodes*, are arranged in a column (on the left), and the nodes in R, the *right nodes*, are arranged in a different parallel column (on the right), in both cases following top-down the total order given by their indices in their own set; the edges are segments connecting the corresponding nodes (see figure 1). Throughout the rest of the paper the pair (G, λ) will denote a bipartite graph G and a prescribed layout λ of it.

Let $b = (b(l_1), b(l_2), \dots, b(l_{|L|}), b(r_1), b(r_2), \dots, b(r_{|R|}))$ be an array of $|L| + |R|$ non-negative integers values. A *non-crossing b-matching* K for (G, λ) is a subset of edges of G with the following two properties: *i*) two non-adjacent edges of K do not have a common coordinate (that is, they do not cross each other in the plane); and *ii*) no more than $b(v)$ edges of K are incident on vertex v for all $v \in L \cup R$. Clearly, a non-crossing b-matching K of $G = (L \cup R, E)$ can be defined with respect to a given layout λ of G, only. In figure 1, two different layouts λ (left) and λ'(right) for a graph G are drawn: for $b = (2, 2, 1, 2, 1, 1, 2)$, the subset $K = \{(l_1, r_3), (l_2, r_4), (l_2, r_4)\} \subseteq E$ is a non-crossing b-matching w.r.t. λ, but not w.r.t. λ'.

Problem $\mathcal{P}$ is formally stated as follows.

Problem 1.1 *$\mathcal{P}$: Given a pair (G, λ), and an array $b = (b(l_1), b(l_2), \dots, b(l_{|L|}), b(r_1), b(r_2), \dots, b(r_{|R|}))$ of integers verifying $b(v) \geq 1$ for all $v \in L \cup R$, find a partition $\langle E_1, E_2, \dots, E_{ncbm(G',\lambda)} \rangle$ of E into the minimum number $ncbm(G, \lambda)$ of non-crossing b-matchings.*

The conditions $b(v) \geq 1$ for all $v \in L \cup R$ are necessary to ensure the problem has feasible solutions.

Among the combinatorial problems related to $\mathcal{P}$ we recall the following ones, which we assume to be all defined on bipartite graphs with n nodes and m edges: maximum cardinality matching [1]; maximum cardinality b-matching; maximum cardinality non-crossing matching (the layout is given; an $O(n \log \log n)$ algorithm is proposed in [5]); maximum cardinality non-crossing b-matching (the layout is given); partition of the edge set into the minimum number of matchings (this problem corresponds to the edge colouring problem, which can be solved in $O(m\sqrt{n})$ time by means of a matching algorithm; see