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Discrete Tomography of Mathematical Quasicrystals: A Primer

Christian Huck, Michael Baake [1]

Fakultät für Mathematik
Universität Bielefeld
Bielefeld, Germany

Barbara Langfeld, Peter Gritzmann, Katja Lord [2]

Zentrum Mathematik
Technische Universität München
München, Germany

Abstract

This text is a report on work in progress. We introduce the class of cyclotomic model sets (mathematical quasicrystals) $\varLambda \subset \mathbb{Z}[\xi_n]$, where $\mathbb{Z}[\xi_n]$ is the ring of integers in the nth cyclotomic field $\mathbb{Q}(\xi_n)$, and discuss the corresponding decomposition, consistency and reconstruction problems of the discrete tomography of these sets. Our solution of the so-called decomposition problem also applies to the case of the square lattice $\mathbb{Z}^2 = \mathbb{Z}[\xi_4]$, which corresponds to the classical setting of discrete tomography.

Keywords: Consistency problem, cyclotomic model set, decomposition problem, discrete tomography, reconstruction problem.

[1] Email:`huck@math.uni-bielefeld.de`
[2] Email:`langfeld@ma.tum.de`

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1 Introduction

Discrete tomography is mainly concerned with the reconstruction of a finite patch of (atomic) positions from projection data, called *X-rays*, along certain rays of parallel lines (or, more generally, from other systems of intersecting sets). In the simplest situation, the positions to be determined form a subset of the square lattice (or, more generally, of $\mathbb{Z}^d$, $d \geq 2$) and this can be considered to be the main and best understood example. In fact, most of the problems in discrete tomography have been studied on the square lattice $\mathbb{Z}^2$ (see [10]), which will be referred to as the classical case. In the longer run, one has to take into account more general classes of sets, or at least significant deviations from the lattice structure. As an intermediate step between periodic and random (or amorphous) sets, we want to investigate the discrete tomography of systems of *aperiodic order*, more precisely, of so-called *mathematical quasicrystals* (or *model sets*), which are commonly accepted to be a good mathematical model for quasicrystalline structures that appear in nature [17].

Here, we restrict ourselves to a well-known class of planar model sets, namely, using the Minkowski representation of algebraic number fields, we introduce for $n \notin \{1, 2, 3, 4, 6\}$ the corresponding class of *cyclotomic model sets* $\varLambda \subset \mathbb{Z}[\xi_n]$, where ξ_n is a primitive nth root of unity (e.g., $\xi_n = e^{\frac{2\pi i}{n}}$). The $\mathbb{Z}$-module $\mathbb{Z}[\xi_n]$ is the ring of integers in the nth cyclotomic field $\mathbb{Q}(\xi_n)$, and with the above restrictions, when viewed as a subset of the plane, is dense. Well-known examples are the planar model sets with N-fold cyclic symmetry associated with the the Ammann-Beenker tiling ($n = N = 8$), the Tübingen triangle tiling ($2n = N = 10$) and the shield tiling ($n = N = 12$). Note that $5, 8, 10$ and 12 are standard cyclic symmetries of genuine planar quasicrystals [17].

We consider the consistency and the reconstruction problem of the discrete tomography of cyclotomic model sets given X-rays in $m \geq 2$ directions and indicate that they are algorithmically solvable in polynomial time if $m = 2$. This extends well-known results from the classical case to the new setting. There are other important results concerning the classical case that can possibly be extended, e.g., the uniqueness results of Gardner and Gritzmann, compare [10, Ch. 4].

Let F be a finite subset of $t + \mathbb{Z}[\xi_n]$, where $n \geq 3$ and $t \in \mathbb{R}^2$. Furthermore, let $o \in \mathbb{Z}[\xi_n] \backslash \{0\}$ be a *module direction* (other directions are not considered for practical reasons) and let $\mathcal{L}_o$ be the set of lines in direction o in the Euclidean